UNIT-I DISCRETE FOURIER TRANSFORM

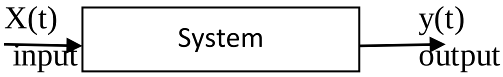
Review of signals and systems, concept of frequency in discrete-time signals, summary of analysis & synthesis equations for FT & DTFT, frequency domain sampling, Discrete Fourier transform (DFT) - deriving DFT from DTFT, properties of DFT - periodicity, symmetry, circular convolution. Linear filtering using DFT. Filtering long data sequences - overlap save and overlap add method. Fast computation of DFT - Radix-2 Decimation-in-time (DIT) Fast Fourier transform (FFT), Decimation-in-frequency (DIF) Fast Fourier transform (FFT). Linear filtering

Introduction:

Signal: Signal is defined as any physical quantity that varies with time, space or any other independent variable.

System:

A system is defined as physical device that generates a response or an output signal, for a given input signal.



where,

y(t) ----> operation on x(t)

y(t) = T[x(t)]

Types of Signal:

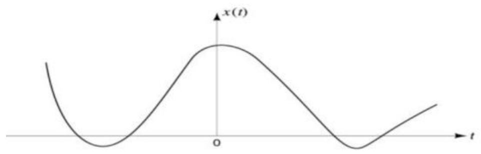
There are two types of signal, that are based on time because the signal basically varies with respect to time, they are,

- Continuous time signal
- Discrete time signal

Continuous time signals (CT):

CT Signals are continuously varied in accordance with the time.

Example: Sine wave, cosine wave etc.

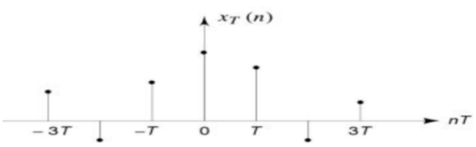


Cosine wave

Discrete time signals (DT):

DT signal is defined as at discrete instants of time are known as discrete-time signals. The discrete-time signals are continuous in amplitude and discrete in time. They are denoted by x(n).

Example:



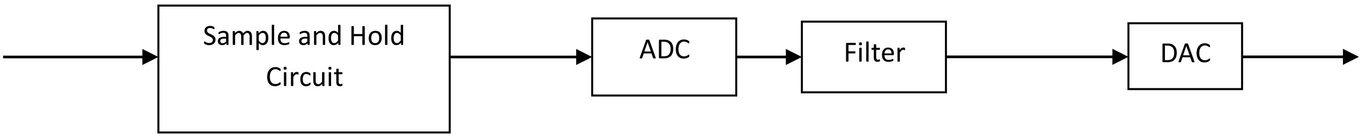
Basic Elements of DSP:

1. Explain briefly Basic Elements of Digital Signal Processing.

Digital signal processing system contains following elements are,

- Analog signal
- Sample and Hold circuit
- ADC (Analog to Digital Converter)
- Filter

- DAC (Digital to Analog Converter)



- Analog signal is nothing but real time signal. It may be speech, video etc.
- Analog signal (continuous –time signal) is given to sample and hold circuit.
- After completing the sampling process, the signal is given to ADC block.
- ADC converts Analog to Digital signal. Then it is given to filter block. This filter removes unwanted noise from the signal. Then the output is processed through the channel.
- In the receiver side (Rx), digital signal is again converted into analog signal. Because analog signal is the real time signal and we can understand only analog signal.

<u>Concepts of frequency in Analog and Digital Signals: (Continuous-time and discrete time signals</u>

2. Discuss the concepts of frequency in Analog and Digital Signals and its Properties.

A simple harmonic oscillation is mathematically described by the following continuous-time sinusoidal signal.

$$xa(t) = A\cos(\Omega t + \theta), -\infty < t < \infty$$

Where,

A →Amplitude of the sinusoid.

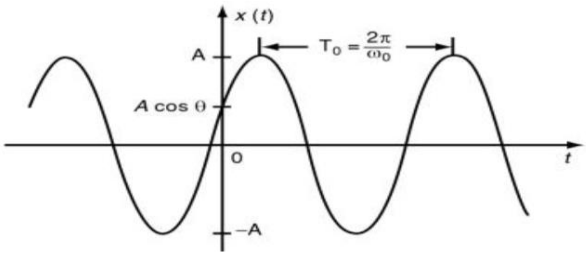
Ω →Frequency in radian/Seconds; $\Omega = 2\pi F$

θ →Phase in radians.

Xa(t) →Analog signal representation.

In terms of F, eqn(1) can be written as,

$$xa(t) = A\cos(2\pi F t + \theta), -\infty < t < \infty$$



$$X(t) = X(t + T)$$

Where,

T →Time period in sec.

t → frequency in Hz.

X(t) → signal represent after the time period T. frequency 'f' satisfies the relationship. $0 \le f \le \infty$

We can prove that, $xa(t) = A\cos(\Omega t + \theta)$

$$= A\frac{1}{2}[e^{j(\Omega t+\theta)} + e^{-j(\Omega t+\theta)}]$$

$$e^{j(\Omega t+\theta)} = \cos(\Omega t + \theta) + j\sin(\Omega t + \theta)$$

$$e^{-j(\Omega t+\theta)} = \cos(\Omega t + \theta) - j\sin(\Omega t + \theta)$$

Add equation (5) & (6),

$e^{j(\Omega t+\theta)} + e^{-j(\Omega t+\theta)} = \cos(\Omega t + \theta) + j\sin(\Omega t + \theta) + \cos(\Omega t + \theta) - j\sin(\Omega t + \theta) = 2\cos(\Omega t + \theta)$

So, $2\cos(\Omega t + \theta) = [\frac{1}{2}[e^{j(\Omega t+\theta)} + e^{-j(\Omega t+\theta)}]]$

Result is $xa(t) = \frac{A}{2}e^{j(\Omega t+\theta)} + \frac{A}{2}e^{-j(\Omega t+\theta)}$

(ii) Discrete- time signals:

A discrete-time sinusoidal signal may be expressed as,

$$x(n) = A\,cos\,(\omega n + \theta), -\infty < n < \infty \text{ ----------> (1)}$$

Where,

n → An integer variable.

A → Amplitude of the sinusoid.

ω -------> Frequency in radians per sample

θ --------> Phase in radians.

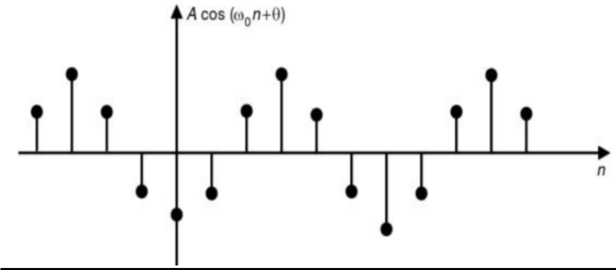
If instead of ω, we use the frequency variable 'f'.

$$\omega = 2\pi f \text{ ---> (2)}$$

Equation (1) becomes,

$$x(n) = A\,cos\,(2\pi f n + \theta), -\infty < n < \infty \text{ --------> (3)}$$

The frequency 'f' has dimensions of cycle per sample.



In contrast to continuous-time sinusoids, the discrete-time sinusoids are characterized by the following properties.

Properties:

- A discrete-time is periodic only if its frequency is a rational number.
- Discrete-time sinusoid whose frequencies are separated by an integer multiple of 2π are identical.
- The highest rate of oscillation in a discrete-time sinusoid is attained. When $(\omega = \pi)$ (or) $(\omega = -\pi)$ or equivalently $(f = \frac{1}{2}\ or = -\frac{1}{2})$

SAMPLING THEOREM:

3. Derive an expression for Sampling theorem and its Reconstruction of *x(t)*. [Nov/Dec-2010][May/June-12][May/june-2015]

Any arbitrary band-limited continuous time signal can be represented in its samples and recovered from its samples taken at equal intervals at the sampling rate of $f_s \geq 2f$ samples/ second. This sampling theorem and reconstruction formula require infinite number of samples. But practically it is not possible to take infinite number of samples.

When the analog frequency is band limited to the range given by the sampling theorem. The digital frequency response is also changed in the same manner.

If $f_s < 2f$ then aliasing problem will occur.

Aliasing effect:

Aliasing is a problem due to interference of information between two band of frequencies. The output due to aliasing is known as an aliased representation of the original signal. This effect should be avoided.

How to avoid aliasing effect?

According to sampling theorem, sampling the continuous time waveform at a high rate is the only one way to avoid aliasing effect.

Nyquist rate:

The sampling rate is generally referred as Nyquist rate.

Proof of sampling theorem:

Consider *x(t)* as input continuous signal. It has finite energy and finite duration. This *x(t)* is band limited signal.

$T_s=1/f_s$ = Sampling period

f_s= Sampling frequency

Impulse function is given as

$$X_\delta(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_S) \text{---------------------} > (1)$$

After sampling input x(t) is represented as $X_\delta(t)$

$$X_\delta(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_S).x(nT_S) \text{-------------------------} > (2)$$

Fourier transform of equation is given as

$$X_\delta(t) = f_S \sum_{n=-\infty}^{\infty} X(f - nf_S) \text{--------------------------} > (3)$$

$$X_\delta(f) = f_S X(f - 0.f_S) + f_S \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} X(f - nf_S) \text{--------} > (4)$$

$$X_\delta(f) = f_S X(f) + f_S \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} X(f - nf_S) \text{-----------------} > (5)$$

Find FT of *x(t)*

If we take the Fourier transform of continuous signal *x(t)*, we will get

$$X(f) = \int_{-\infty}^{\infty} x(t).e^{-j2\pi ft} dt \quad \text{-----------------------} > (6)$$

If we write above equation for discrete signal put $t=nT_S$

$$X_\delta(t) = \sum_{n=-\infty}^{\infty} x(nT_S).e^{-j2\pi f.nT_S} \text{-------------------} > (7)$$

Rearrange equation (5)

$$X(f) = \frac{1}{f_S} X_\delta(f) - \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} X(f - nf_S) \text{------------} > (8)$$

$$X(f) = \frac{1}{f_S} X_\delta(f) \text{--------------------------} > (9)$$

If frequency lies between –f to +f

Put eqn (7) in (9)

$$X(f) = \frac{1}{f_S}\left[\sum_{n=-\infty}^{\infty} x\left(n\frac{1}{f_S}\right) e^{-j2\pi f.n\frac{1}{f_S}}\right] \text{-------------} > (10)$$

Reconstruction of *x(t)*:

x(t) can be reconstructed from equation (10) and put $f_S=2f$

$$x(f) = \frac{1}{2f}\left[\sum_{n=-\infty}^{\infty} x\left(\frac{n}{2f}\right) e^{-j2\pi n\frac{1}{2f}}\right] \text{----------------} > (11)$$

Take inverse Fourier transform of above equation

$$x(t) = \int_{-f}^{f}\left[\frac{1}{2f}\sum_{n=-\infty}^{\infty} x\left(\frac{n}{2f}\right) e^{\frac{-j\pi n}{f}}\right] e^{j2\pi ft}.df \text{-----------} > (12)$$

$$x(t)=\sum_{n=-\infty}^{\infty} x\left(\frac{n}{2f}\right)\frac{1}{2f}\int_{-f}^{f}\left[e^{j2\pi f\left(t-\frac{n}{2f}\right)}\right].\mathrm{df}$$

(Order of summation and integration is interchanged)

$$x(t)=\sum_{n=-\infty}^{\infty} x\left(\frac{n}{2f}\right)\left(\frac{\sin(2\pi ft-n\pi)}{(2\pi ft-n\pi)}\right)$$

In above equation put $\sin c\theta=\frac{\sin\pi\theta}{\pi\theta}$ formula, which is the sin c function

$$\frac{\sin(2\pi ft-n\pi)}{(2\pi ft-n\pi)}=\sin c(2ft-n)$$

Now,

$$x(t)=\sum_{n=-\infty}^{\infty} x\left(\frac{n}{2f}\right)\sin c(2ft-n)$$

The above equation is also known as interpolation formula and it expand as,

$$x(t)=x(0).\sin c(2\pi ft)+x\left(\pm\frac{1}{2f}\right)\sin c(2ft\pm 1)+........$$

Challenge: 1

Consider the analog signal $xa(t) = 3cos100\pi t$.

(a) Determine the minimum sampling rate required to avoid aliasing.

(b) Find the nyquist rate and Nyquist interval.

(c) Find the folding frequency.

Given: Analog signal $xa(t) = 3cos100\pi t$.

General equation $x(t) = Acos\omega t$.

Compare with given equation, A= 3; $\omega = 100\pi$

a) **Determine the minimum sampling rate required to avoid aliasing.**

$$fs \geq 2\omega$$
$$2\pi f = \omega = 100\pi$$
$$f = 50$$

This is the maximum frequency f_{max} (or) ω

$$\omega = 50$$

Minimum sampling rate $fs = 2\,\omega = 2*50 = 100Hz$

b) **Find the Nyquist rate and Nyquist interval.**

Nyquist rate = $2\,\omega = 2*50 = 100Hz$.

Nyquist Interval = $\frac{1}{2\,\omega} = \frac{1}{2*50} = \frac{1}{100} secs$

c) **Find the folding frequency.**

$$folding\ frequency(Ff) = \frac{fs}{2} = \frac{sampling\ frequency}{2} = \frac{100}{2} = 50Hz$$

Half of the sampling frequency = $50Hz$

<u>Discrete – Time Signals:</u>

- Representation of Discrete – Time Signals:
- Elementary of Discrete – Time Signals:
- Classification of Discrete – Time Signals:
- Operation of Signals:

Representation of Discrete – Time Signals:

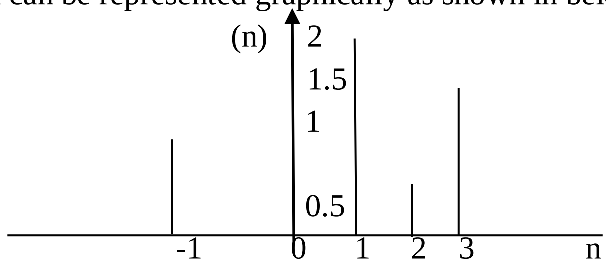
4. Explain the different representation of Discrete-time signals. [Nov/Dec-13]

There are different types of representation for discrete-time signals. They are

- Graphical representation
- Functional representation
- Tabular representation
- Sequence representation

Graphical representation:

Let us consider a signal x(n) with values $x(-1)=1; x(0)=2; x(1)=2; x(2)=0.5$ and $x(3)=1.5$. This discrete-time signal can be represented graphically as shown in below.



Functional Representation:

The discrete-time signal can be represented using functional representation is below.

$$x(n) = \begin{cases} 1 \ for \ n = -1 \\ 2 \ for \ n = 0,1 \\ 0.5 \ for \ n = 2 \\ 1.5 \ for \ n = 3 \\ 0 \ otherwise \end{cases}$$

Tabular Representation:

The discrete-time signal can also be represented as,

n	-1	0	1	2	3
x(n)	1	2	2	0.5	1.5

Sequence Representation:

A finite duration sequence with time origin (n = 0) indicated by the symbol $\uparrow$ is represented as

$$x(n) = \{1, \underset{\uparrow}{2}, 2, 0.5, 1.5\}$$

An infinite duration sequence can be represented as,

$$x(n) = \{...0.2, \underset{\uparrow}{1}, -1, 3, 2........\}$$

A finite duration sequence that satisfies the condition $x(n) = 0$ for n<0 can be represented as,

$$x(n) = \{2,4,6,8-3\}$$

Elementary of Discrete – Time Signals:

5. Explain the different representation of Discrete-time signals.

There are different types of elementary of discrete-time signals are,

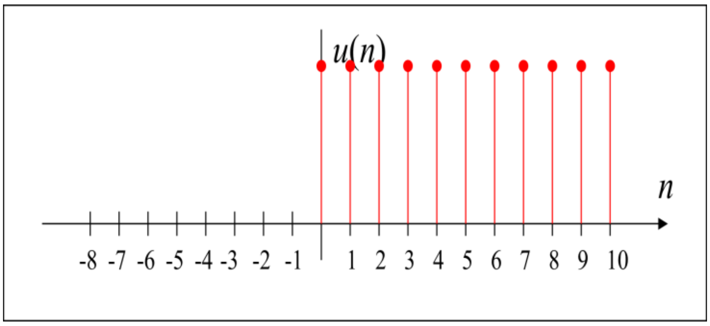
- Unit step sequence
- Unit ramp sequence
- Unit-sample sequence
- Exponential sequence
- Sinusoidal sequence
- Complex exponential sequence.

Unit step sequence:

The unit step sequence is defined as

$$u(n) = \begin{cases} 1 \; for \; n \geq 0 \\ 0 \; for \; n < 0 \end{cases}$$

The graphical representation of u(n) is shown in figure.

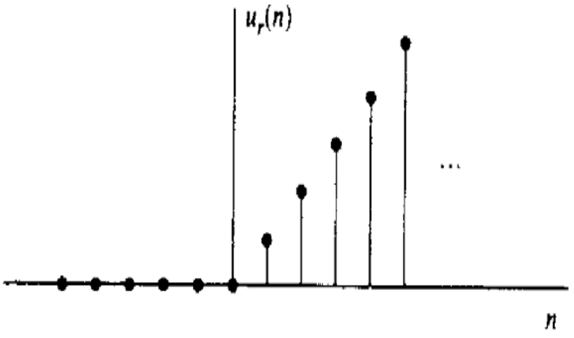


Unit ramp sequence:

The unit ramp sequence is defined as

$$r(n) = \begin{cases} n \; for \; n \geq 0 \\ 0 \; for \; n < 0 \end{cases}$$

The graphical representation of r(n) is shown in figure

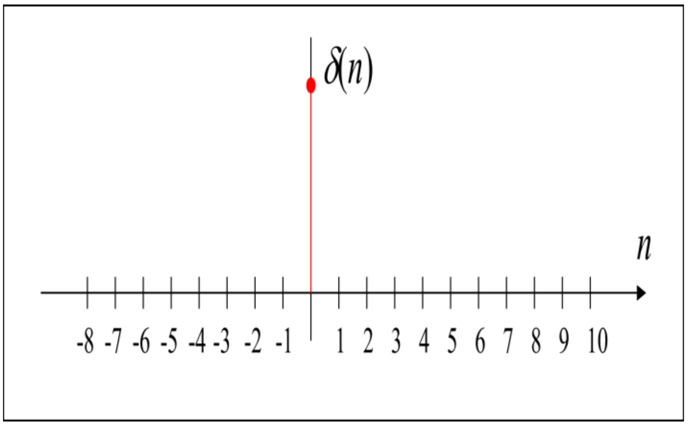


Unit-sample sequence (unit impulse sequence):

The unit-sample sequence is defined as,

$$\delta(n) = \begin{cases} 1 \; for \; n = 0 \\ 0 \; for \; n \neq 0 \end{cases}$$

The graphical representation of $\delta(n)$ is shown in figure.



The unit impulse unction has the following properties.

$\delta(n) = u(n) - u(n-1)$-----------------> (1)

$u(n) = \sum_{k=-\infty}^{n} \delta(k)$-----------------------> (2)

$\sum_{n=-\infty}^{\infty} x(n)\delta(n-no) = x(no)$---------> (3)

Exponential sequence:

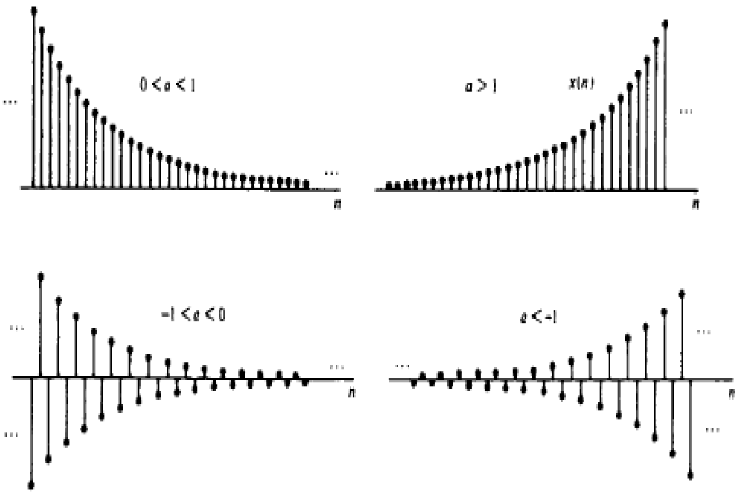
The exponential signal is a sequence of the form

$$x(n) = a^n \; for \; all \; n$$

Different types of discrete-time exponential signals.

- When the value of a > 1, the sequence grows exponentially and
- When the values is 0 < a < 1, the sequence decay exponentially.

- When a < 0, the discrete-time exponential signal takes alternating signs.



Sinusoidal signal:

The discrete-time sinusoidal signal is given by,

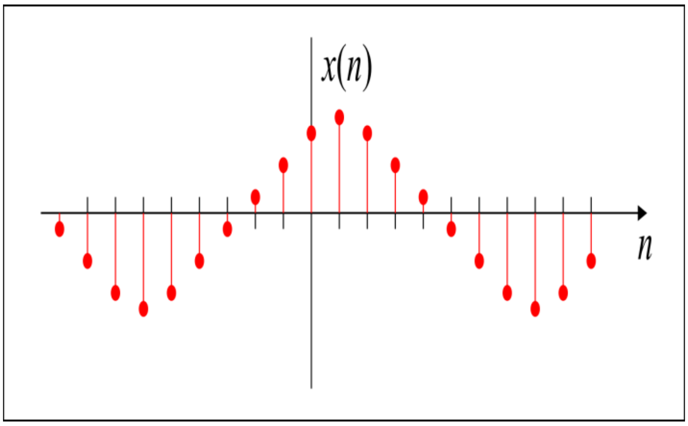
$x(n) = A\cos(\omega o n + \phi)$ ---------> (1)

Where, ωo - is the frequency (in radians per sample) and ϕ is the phase (in radians).

Using euler's identity, we can write

$A\cos(\omega on + \varphi) = \frac{A}{2}e^{j\varphi}e^{j\omega on} + \frac{A}{2}e^{-j\varphi}e^{-j\omega on}$-------> (2)

Since $\left|e^{j\omega on\varphi}\right|^2 = 1$, the energy of the signal is infinite and the average power of the signal is 1.

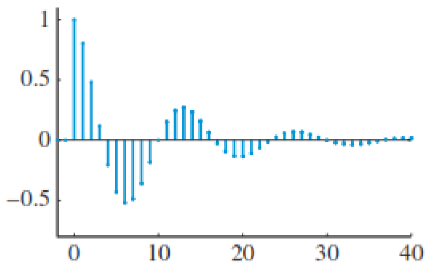


Complex Exponential signal:

The discrete-time complex exponential signal is given by

$$x(n) = a^n e^{j(\omega on + \varphi)}$$

$= a^n\cos(\omega on + \varphi) + ja^n \sin(\omega on + \varphi)$------ →(1)



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Classification of Discrete-time signals:

There are different types of classification of discrete-time signals are,

- Energy signals and power signals.

- Periodic and Aperiodic signals
- Symmetric(even) and ant-symmetric(odd) signals.
- Causal and non-causal signals.

Energy signals and power signals:

Energy signal:

A discrete-time signal x(n) the energy ‘E’ is defined as,

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

Power signal:

The average power of a discrete-time signal x(n) is defined as,

$$P = \lim_{n\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

Note:

- A signal is energy signal, if an only if the total energy of the signal is finite. For an energy signal P=0.
- A signal is power signal, if the average power of the signal is finite. For power signal E=∞.
- The signals that do not satisfy above properties are neither energy nor power signals.

Challenge 1:

Determine the values of power and energy of the following signals. Find whether signals are power, energy or neither energy nor power signals.

(i) $\boldsymbol{x(n) = \left(\frac{1}{3}\right)^n u(n)}$ **[May/June-2016]**

(ii) $\boldsymbol{x(n) = e^{j\left(\frac{\pi}{2}n+\frac{\pi}{4}\right)}}$

Solution:

Given:

(i) signal $x(n) = \left(\frac{1}{3}\right)^n u(n)$

To find Energy signal of x(n)

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$= \sum_{n=-\infty}^{\infty} \left[(\frac{1}{3})^n \right]^2 u(n)$$

$$= \sum_{n=0}^{\infty} \left[(\frac{1}{3})^n \right]^2 \qquad \because u(n) = 1 \; for n \geq 0 \quad = 0 \; for n < 0 \qquad \therefore 1 + a + a^2 + ...\infty = \frac{1}{1-a}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{9}\right)^n$$

$$= \frac{1}{1-9} = \frac{9}{8}$$

To find Power of x(n) :

$$P = \lim_{n\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

$$= \lim \frac{1}{2N+1}\sum_{n=-N}^{N}\left|\left(\frac{1}{3}\right)^n\right|^2 u(n)$$

$$= \lim \frac{1}{2N+1}\sum_{n=0}^{N}\left(\frac{1}{9}\right)^n$$

$$= \lim \frac{1}{2N+1}\left[\frac{1-\left(\frac{1}{9}\right)^{N+1}}{1-\frac{1}{9}}\right]$$

$$= 0$$

Result: The energy is finite and power is zero. Therefore, the signal is an energy signal.

(i) $\boldsymbol{x(n) = e^{j(\frac{\pi}{2}n+\frac{\pi}{4})}}$

Solution:

Given: signal (ii) $x(n) = e^{j(\frac{\pi}{2}n+\frac{\pi}{4})}$

To find Energy of x(n):

$$E = \sum_{n=-\infty}^{\infty}|x(n)|^2$$

$$= \sum_{n=-\infty}^{\infty}\left|e^{j\left(\frac{\pi}{2}n+\frac{\pi}{4}\right)}\right|^2 u(n)$$

$$= \sum_{n=-\infty}^{\infty}\left|e^{j\left(\frac{\pi}{2}n+\frac{\pi}{4}\right)}\right|^2 \qquad \because \left|e^{j(\omega+\theta)}\right| = 1$$

$$= \sum_{n=-\infty}^{\infty} 1 = \infty$$

To find power of x(n):

$$P = \lim_{n\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N}|x(n)|^2$$

$$= \lim \frac{1}{2N+1}\sum_{n=-N}^{N}\left|e^{j\left(\frac{\pi}{2}n+\frac{\pi}{4}\right)}\right|^2$$

$$= \lim \frac{1}{2N+1}\sum_{n=-N}^{N}1 \qquad \because \sum_{n=-N}^{N}1 = 2N+1$$

$$= \lim \frac{1}{2N+1}(2N+1)$$

$$= 1$$

Result: The energy is infinite and power is finite. Therefore, the signal is power signal.

H.W: Find whether the signals are power, energy or neither energy nor power signals.

(i) $\cos(\omega o n)u(n)$ **Ans: Power signal**

(ii) $u(n+2)-u(n-2)$ **Ans: Energy signal**

(iii) **(iii)** $\boldsymbol{x(n) = sin(\frac{\pi}{4}n)}$ **Ans : Power signal**

(iv) $\boldsymbol{x(n) = e^{2n}u(n)}$ **Ans : Power nor energy signal**

Periodic and Aperiodic Signals:

Challenge 1:

Determine whether the signal is periodic or not. If the signal is periodic, find the fundamental period.

(i) $x(n)=e^{j6\pi n}$ **(ii)** $x(n)=e^{j\frac{3}{5}\left(n+\frac{1}{2}\right)}$ **(iii)** $x(n)=\cos\left(\frac{2\pi}{3}\right)n$ **(iv)** $x(n)=\cos\frac{\pi}{3}n+\cos\frac{3\pi}{4}n$

Solution:

Given signal: $x(n)=e^{j6\pi n}$. $\boldsymbol{\omega o = 6\pi}$

The fundamental frequency is multiple of π. Therefore, the signal is periodic.

$$N=\left(\frac{2\pi}{\omega o}\right)$$

$$=\left(\frac{2\pi}{6\pi}\right)$$ Therefore the fundamental period is 3.

$$N=3$$

(ii) $x(n)=e^{j\frac{3}{5}\left(n+\frac{1}{2}\right)}$

Solution:

Given Signal: $x(n)=e^{j\frac{3}{5}\left(n+\frac{1}{2}\right)}$; $\omega o = \frac{3}{5}$, which is not a multiple of π. Therefore, the signal is Aperiodic.

H.W.Determine the fundamental period of the following signals, if they are priodic.

(i) $\boldsymbol{x(n) = sin(\frac{\pi n}{4})}$ **Ans:Periodic with N=8**

(ii) $\boldsymbol{x(n) = e^{j2n}}$ **Ans:Aperiodic**

(iii) $\boldsymbol{x(n) = cos\frac{\pi}{4}n + cos2n}$ **Ans:Aperiodic**

(iii) $x(n)=\cos\left(\frac{2\pi}{3}\right)n$

Ans : Periodic with N=3

(iv) $x(n)=\cos\frac{\pi}{3}n+\cos\frac{3\pi}{4}n$

Ans : Periodic with N=24

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Symmetric (even) and antisymmetric(odd) signals:

Symmetric:

A discrete-time signal x(n) is said to be a symmetric (even) signal, if it satisfies the condition.

$\boldsymbol{x(-n) = x(n) for\ all\ n}$------> **(1)**

$$\boldsymbol{xe(n) = \frac{1}{2}[x(n) + x(-n)]}$$

Example: $\cos\omega n$

Antisymmetric:

The signal is said to be an odd signal if it satisfies the condition.

$\boldsymbol{x(-n) = -x(n) for\ all\ n}$--------> **(2)**

$$\boldsymbol{xo(n) = \frac{1}{2}[x(n) - x(-n)]}$$

Example: $\sin\omega n$

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Causal and Non-causal Signals:

A signal is said to be causal, if its value is zero for n < 0. Otherwise the signal is non-causal.

Basic operation on Signals:

6. Explain the basic operation on Signals.

Signal processing is a group of basic operations applied to an input signal resulting in another signal as the output.

- The mathematical transformation from one signal to another is represented as

$y(n) = T[x(n)]$

The basic sets of operations are,

1.Shifting
2. Time reversal
3. Time scaling
4. Scalar multiplication
5 Signal multiplier
6. Signal addition

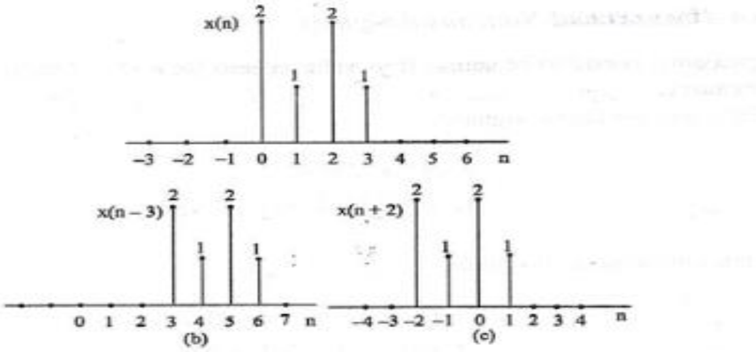
Shifting:

- The shift operation takes the input sequence and shifts the values by an integer increment of the independent variable.
- The Shifting may delay or advance the sequences in time. Mathematically this can be represented as y(n) = x(n - k)

Where x(n) is the input and y(n) is the output.

- ✓ If k is positive. the shifting delays the sequence.
- ✓ If k is negative. the shifting advances the sequence.

Example: The signal x(n - 3) is obtained by shifting x(n) right by 3 units of time. The result is shown in Figure. On the other hand, the signal x(n+2) is obtained by shifting x(n) left by two units of time.

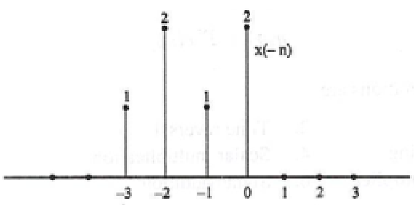


(a) Discrete time signal (b) delayed version (c) advanced version

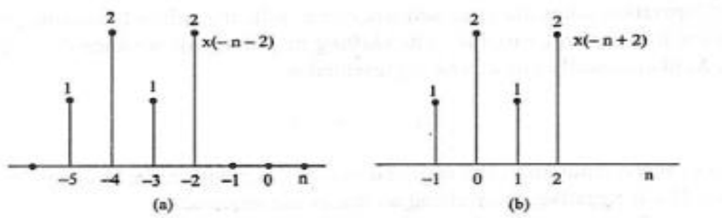
Time reversal:

- The time reversal of sequence x(n) can be obtained by folding the sequence about n = 0.It is denoted as x(-n). For the signal x(n) shown in figure, the x(-n) is given in figure.

Example: The signal x(-n + 2) is x(-n) delayed by two units of time and x(-n- 2) is x(-n) advanced by two units of time. The graphical representation of x(-n - 2) and x(-n + 2) are shown in figure.



Timely reversal of Signal $x(n)$



(a)Shifted version of the signal $x(-n)$ (a) advance (b) delay

Time Scaling:

This is accomplished by replacing 'n' by λn in the sequence x(n). Let x(n) is a sequence shown in figure.

If λ = 2 we get a new sequence

y(n) = x(2n).

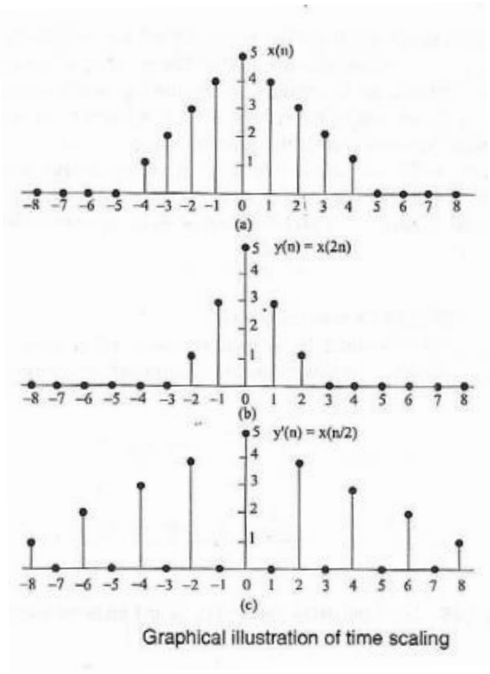
We can plot the sequence y(n) by substituting different values for n.

For $n = -1; y(-1) = x(-2) = 3$

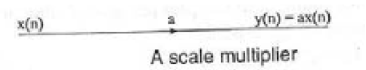
$y(0) = x(0) = 5$

Similarly, $y(1) = x(2) = 3$

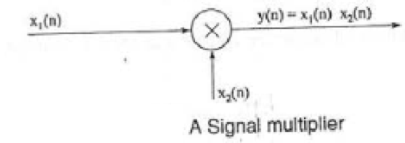
$y(2) = x(4) = 1$



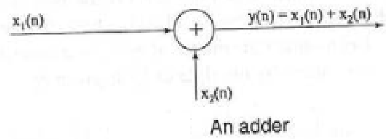
Scalar Multiplication:
A scalar multiplier is shown in below. Here, the signal x(n) is multiplied by a scale factor a.



For example if $x(n) = \{1,2,1,-1\} and\ a = 2$. then the signal $ax(n) = \{2,4,2,-2\}$
Signal Multiplier:
The multiplication of two signal sequence to form another sequence.



For example, if $x1(n) = \{-1,2,-3,-2\} \quad and x2(n) = \{1,-1,-2,1\}$
Then, $x1(n).x2(n) = \{-1,-2,6,-2\}$
Addition Operation:
Two signals can be added by using an adder shown in below.



For example, if
$x1(n) = \{1,2,3,4\} and$
$x2(n) = \{4,3,2,1\}$
Then, $x1(n) + x2(n) = \{5,5,5,5\}$

Classification of Discrete-time systems:

7. Discuss the various Classifications of Discrete-time systems. [May/June-2013][Nov/Dec-13][April/May-14][Nov/Dec-14]

Discrete-time systems are classified according to their general properties and characteristics. They are,
- Static and Dynamic systems
- Causal and Non-causal systems
- Linear and Non-linear systems
- Time variant and Time-Invariant systems
- Stable and Unstable systems.

Static and Dynamic systems

Static systems:

- A Discrete-time system is called Static or Memory less if its output at any instant 'n' depends on the input samples at the same time, but not on past or future samples of the input. The output at any instant depends on the input at that instant.

Example:
$y(n) = ax(n)$

Dynamic systems:

- A Discrete-time system is said to be dynamic or to have memory, if the output of y(n) depends on past or future samples of the input. The output depends on past values of input. It requires memory.

Example:
$y(n) = x(n-1) + x(n-2)$

Causal and Non-causal Systems:

Causal system:

A system is said to be causal if the output of the system at any time n [i.e y(n)] depends only on present and past inputs [i.e., x(n), x(n-1, x(n-2)…)]but does not depend on future inputs [i.e., x(n+1, x(n+2)…)]. In mathematical terms, the output of a casual system satisfies an equation of the form

$$y(n) = F[x(n), x(n-1), x(n-2), \dots.]$$

where F[.] is some arbitrary function.

Example:
$y(n) = x(n) + x(n-1)$

Non-Causal systems:

If the output of a system depends on future inputs, the system is said to be non-causal or anticipatory.

Example:
$y(n) = x(2n)$

Linear system and Non-linear systems:

Linear system:

A linear system is the one that satisfies the superposition principle. Superposition principle states that the response of the system to a weighted sum of signals be equal to the corresponding weighted sum of responses (output) of the system to each of the individual input signals.

A system is linear if and only if

$$\mathcal{T}[a_1 x_1(n) + a_2 x_2(n)] = a_1 \mathcal{T}[x_1(n)] + a_2 \mathcal{T}[x_2(n)]$$

where $x_1(n)$ and $x_2(n)$ are arbitrary input sequence and a_1 and a_2 are any arbitrary constants .

Example:
$y(n) = n\, x(n)$

Non-linear systems:

A system does not satisfy the superposition principle is called non- linear system.

$$\mathcal{T}[a_1 x_1(n) + a_2 x_2(n)] \neq a_1 \mathcal{T}[x_1(n)] + a_2 \mathcal{T}[x_2(n)]$$

Example:

$y(n)=Ax(n) + B$

Time Variant and Time-Invariant systems:

Time invariant (Shift Invariant)system:

A relaxed system $\mathcal{T}$ is time invariant or shift invariant if and only if

$$x(n) \xrightarrow{\mathcal{T}} y(n)$$

Implies that

$$x(n-k) \xrightarrow{\mathcal{T}} y(n-k)$$

For every input signal x(n) and every time shift k.

In other words, A system is said to be time-invariant or shift invariant if the characteristics of the system do not change with time.

i.e $y(n,k) = y(n-k)$

Example:

Differntiator

Time variant (Shift variant)system:

A system is said to be time-variant or shift variant if the characteristics of the system changes with time.

$y(n,k) \neq y(n-k)$

Stable and Unstable systems:

Stable system:

An arbitrary relaxed system is said to be bounded input-bounded output (BIBO) stable if and only if every bounded input produces a bounded output.

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

Unstable System:

If, for some bounded input sequence x(n), the output is unbounded(infinite), the system is classified as unstable.

****

Challenge 1: find whether the following systems are static or dynamic.

(i) $y(n) = x(n)x(n-1)$ **(ii)** $y(n) = x^2(n) + x(n)$

Solution:

Given: Output system $y(n) = x(n)x(n-1)$

The output y(n) depends on the past input. Therefore the system is dynamic.

(ii) $y(n) = x^2(n) + x(n)$

Solution:

Given: output system $y(n) = x^2(n) + x(n)$

The output y(n) depends on the present input only. Therefore the system is static.

****

H.W: check whether the following systems are static or dynamic.

(i) $y(n) = x(2n)$ Ans: Dynamic

(ii) $y(n) = x^2(n)$ Ans: Static

****

Challenge 2: Test whether the following systems are causal or non-causal.

(i) $y(n) = x(n) + \frac{1}{x(n-1)}$ **(ii)** $y(n) = x(n^2)$

Solution:

Given: output of the system $y(n) = x(n) + \frac{1}{x(n-1)}$

For n=-1; $y(-1) = x(-1) + \frac{1}{x(-2)}$

For n=0; $y(0) = x(0) + \frac{1}{x(-1)}$

For n=1; $y(1) = x(1) + \frac{1}{x(0)}$

For all values of ‘n’, the output depends on present and past inputs. Therefore, the system is causal.

(ii) $y(n) = x(n^2)$

Solution:

Given: output of the system $y(n) = x(n^2)$

For n= -1; $y(-1) = x(1)$

For n=0; $y(0) = x(0)$

For n= 1; $y(1) = x(1)$

For all values of 'n', the output depends on future inputs. So, the system is Non-causal.

H.W: **Test whether the following systems are causal or non-causal.**

(i) $y(n) = Ax(n) + B$ Ans: Causal

(ii) $y(n) = ax(n) + bx(n-1)$ Ans: Causal.

Challenge 3: Determine if the system described by the following input-output equations is linear or non-linear.[May/June-2016]

(i) $y(n) = x(n) + \frac{1}{x(n-1)}$ **(ii)** $y(n) = nx(n)$

Solution:

Given: output of the system $y(n) = x(n) + \frac{1}{x(n-1)}$

For two input sequences $x1(n) and\ x2(n)$ the corresponding outputs are,

$$y1(n) = T[x1(n)] = x1(n) + \frac{1}{x1(n-1)} \text{ ---------> (1)}$$

$$y2(n) = T[x2(n)] = x2(n) + \frac{1}{x2(n-1)} \text{ ---------> (2)}$$

The output due to weighted sum of input is

$$y3(n) = T[a1x1(n) + a2x2(n)] = a1x1(n) + \frac{1}{a1x1(n-1)} + a2x2(n) + \frac{1}{a2x2(n-1)} \text{ -----> (3)}$$

The linear combination of the two output is,

$$a1y1(n) + a2y2(n) = a1x1(n) + \frac{a1}{x1(n-1)} + a2x2(n) + \frac{a2}{x2(n-1)} \text{ ---------> (4)}$$

$\therefore$ equation (3) and (4) are not equal. So, the system is non-Linear.

(ii) $y(n) = nx(n)$

Solution:

Given: output of the system $y(n) = nx(n)$

For two input sequences $x1(n) and\ x2(n)$ the corresponding outputs are,

$$y1(n) = T[x1(n)] = nx1(n) \text{ ---------> (1)}$$

$$y2(n) = T[x2(n)] = nx2(n) \text{ ---------> (2)}$$

The output due to weighted sum of input is

$$y3(n) = T[a1x1(n) + a2x2(n)] = a1nx1(n) + a2nx2(n) \text{ ---------> (3)}$$

The linear combination of the two output is,

$$a1y1(n) + a2y2(n) = \left[a1nx1(n) + a2nx2(n)\right] \text{ ---------> (4)}$$

$\therefore$ equation (3) and (4) are equal. So, the system is Linear.

****

H.W: check whether the system is linear or not.

(i) $y(n) = 2x(n) + \frac{1}{x(n-1)}$ **Ans: Non-linear**

(ii) $y(n) = nx^2(n)$ **Ans: Non-linear**

(iii) $y(n) = x^2(n)$ **Ans: Non-linear**

****

Challenge 4: Determine if the following systems are time-invariant or time variant.

(i) $y(n) = x(n) + x(n-1)$ **(ii)** $y(n) = x(-n)$

Solution:

Given: output of the system $y(n) = x(n) + x(n-1)$

If the input is delayed by 'k' units in time, we have

$$y(n,k) = x(n-k) + x(n-k-1) \text{ ---------> (1)}$$

If the output is delayed by 'k' units in time, then (n-> n-k)

$$y(n-k) = x(n-k) + x(n-k-1) \text{---} > (2)$$

Here, $y(n,k) = y(n-k)$

Therefore, the system is time-invariant.

(ii) $y(n) = x(-n)$

Solution:

Given: output of the system $y(n) = x(-n)$

If the input is delayed by 'k' units in time, we have

$$y(n,k) = x(-n-k) \text{---} > (1)$$

If the output is delayed by 'k' units in time, then (n-> n-k)

$$y(n-k) = x(-n+k) \text{---} > (2)$$

Here, $y(n,k) \neq y(n-k)$

Therefore, the system is time-variant.

****

H.W: Determine if the following systems are time invariant.

(i) $y(n) = x\left(\frac{n}{2}\right)$ **Ans: Time-variant**

(ii) $y(n) = nx^2(n)$ **Ans: Time-Variant**

****

Challenge 5: Test the stability of the system whose impulse response $h(n) = \left(\frac{1}{2}\right)^n u(n)$

Solution:

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

$$= \sum_{n=-\infty}^{\infty} \left| \left(\frac{1}{2}\right)^n u(n) \right|$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$$

$$= 1 + \frac{1}{2} + \frac{1}{4} \text{..........}\infty$$

$$= \frac{1}{1-\frac{1}{2}}$$

$$= 2 < \infty$$

Hence the system is stable.

****

Challenge 6: Test if the following systems are stable or not.

(i) $y(n) = x(n)u(n)$ **(ii)** $y(n) = x(n) + nx(n+1)$

Solution:

Given: output of the system $y(n) = x(n)u(n)$

If the input x(n) is bounded, $|x(n)| \leq M$ then

$|y(n)| = |x(n)||u(n)| \leq M$

That is $|y(n)| \leq M$. hence the system is stable.

(ii) $y(n) = x(n) + nx(n+1)$

Solution:

Given: output of the system $y(n) = x(n)u(n)$

If the input x(n) is bounded, $|x(n)| \leq M$ then

$|y(n)| = |x(n) + nx(n+1)|$

The output increases with increasing n. hence the bounded input produce unbounded output. Hence the system is unstable.

****

H.W: Test if the following systems are stable or not.

(i) $y(n) = sign|x(n)|$ **Ans: Stable**

(ii) $y(n) = x(2n)$ **Ans: Stable**

(iii) $y(n) = e^{x(n)}$ **Ans: Stable**

(iv) $y(n) = ax(n) + b$ **Ans: Stable**

(v) $y(n) = x(-n)$ **Ans: Stable**

(vi) $y(n) = x(n)\cos(\omega o n)$ **Ans: Stable**

(vii) $y(n) = \sum_{k=n-no}^{n+no} x[k]$ **Ans: Stable**

****

<u>DFT (Discrete-Fourier Transform)</u>

The sequence X(k) is called the N-point DFT of x(n). These coefficients are related to x(n) as follows:

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-\frac{j2\pi kn}{N}}, \qquad k = 0, 1 \dots N-1$$

The N-point IDFT of the sequence X(k) is

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) e^{\frac{j2\pi kn}{N}}, \qquad n = 0, 1 \dots N-1$$

Challenge 1.

$Compute The DFT of the four-po\text{int} sequences x(n) = \{0,1,2,3\}.$

$Given x(n) = \{0,1,2,3\}$

$$X(\text{k}) = \sum_{n=0}^{\infty} x(n) e^{-j\pi kn/N} \qquad k = 0,1,2,3$$

$$X(0) = 0 + 1 + 2 + 3 = 6$$

$$X(1) = \sum_{n=0}^{3} x(n) e^{-j\pi kn/2}$$

$$= 0 + 1(-j) + 2(-1) + 3(j) = -2 + \text{j}2$$

$$X(2) = \sum_{n=0}^{3} x(n) e^{-j\pi kn}$$

$$= 0 + 1(-1) + 2(1) + 3(-1) = -2$$

$$X(3) = \sum_{n=0}^{3} x(n) e^{-j\pi kn/2}$$

$$= 0(1) + 1(j) + 2(-1) + 3(-j) = -2 - j2$$

$$X(k) = \{6, -2 + j2, -2, -2 - j2\}$$

****

Challenge 2:Determine the 8-point DFT of the sequence x(n)={1,1,1,1,1,1,0,0}.[Nov/Dec -2010][April/May-2011]

Solution:

$$X(\text{k}) = \sum_{n=0}^{N-1} x(\text{n})\, \text{e}^{-j2\pi kn/N} k = 0,1,\dots N-1$$

For N=8

$$X(\mathrm{k}) = \sum_{n=0}^{7} x(\mathrm{n})\,\mathrm{e}^{-j\pi kn/4} k = 0,1,2....N-1$$

For k=0

$$X(0) = \sum_{n=0}^{7} x(\mathrm{n})$$

X(0) = x(0)+x(1)+x(2)+x(3)+x(4)+x(5)+x(6)+x(7)
= 1+1+1+1+1+1+0+0
= 6

For k=1

$$X(1) = \sum_{n=0}^{7} x(\mathrm{n})\,\mathrm{e}^{-j\pi n/4}$$

X(1) = x(0) +x(1) $e^{-j\pi/4}$+x(2) $e^{-j\pi/2}$+x(3) $e^{-j3\pi/4}$+x(4) $e^{-j\pi}$+x(5) $e^{-j5\pi/4}$+x(6) $e^{-j3\pi/2}$+x(7) $e^{-j7\pi/4}$
= 1+0.707-j0.707-j-0.707-j0.707-1-0.707+j0.707 => -0.707-j1.707

For k=2

$$X(2) = \sum_{n=0}^{7} x(\mathrm{n})e^{-j\pi n/2}$$

X(2) = x(0)+x(1) $e^{-j\pi/2}$+x(2) $e^{-j\pi}$+x(3) $e^{-j3\pi/2}$+x(4) $e^{-j2\pi}$+x(5) $e^{-j5\pi/2}$+x(6) $e^{-j3\pi}$+x(7) $e^{-j7\pi/2}$
= 1-j-1+j+1-j
= 1-j

For k=3

$$X(3) = \sum_{n=0}^{7} x(\mathrm{n})e^{-j3\pi n/4}$$

X(3) = x(0)+x(1) $e^{-j3\pi/4}$+x(2) $e^{-j3\pi/2}$+x(3) $e^{-j9\pi/4}$+x(4) $e^{-j3\pi}$+x(5) $e^{-j15\pi/4}$+x(6) $e^{-j9\pi/4}$ +x(7) $e^{-j21\pi/4}$
= 1-0.707-j0.707+j+0.707-j0.707-1+0.707+j0.707
= 0.707 + j0.293

For k=4

$$X(4) = \sum_{n=0}^{7} x(\mathrm{n})\,\mathrm{e}^{-j\pi n}$$

X(4) = x(0)+x(1) $e^{-j\pi}$ +x(2) $e^{-j\pi 2}$ +x(3) $e^{-j\pi 3}$ +x(4) $e^{-j\pi 4}$ +x(5) $e^{-j\pi 5}$ +x(6) $e^{-j\pi 6}$ +x(7) $e^{-j\pi 7}$
= 1-1+1-1+1-1 = 0

For k=5

$$X(5) = \sum_{n=0}^{7} x(\mathrm{n})e^{-j5\pi n/4}$$

X (5) = x(0)+x(1) $e^{-j5\pi/4}$+x(2) $e^{-j5\pi/2}$+x(3) $e^{-j5\pi n/4}$+x(4) $e^{-j5\pi}$+x(5) $e^{-j25\pi/4}$+x(6) $e^{-j15\pi/2}$ +x(7) $e^{-j35\pi/4}$
= 1-0.707+j0.707-j+0.707+j0.707-1+0.707-j0.707
= 0.707-j0.293

For k=6

$$X(6) = \sum_{n=0}^{7} x(\mathrm{n})e^{-j3\pi n/2}$$

X(6) = x(0)+x(1) $e^{-j3\pi/2}$+x(2) $e^{-j3\pi}$+x(3) $e^{-j9\pi/2}$+x(4) $e^{-j6\pi}$+x(5) $e^{-j15\pi}$+x(6) $e^{-j9\pi}$+x(7) $e^{-j21\pi/2}$
= 1+j-1-j+1+j => = 1+j

For k=7

$$X(7) = \sum_{n=0}^{7} x(\mathrm{n})e^{-j7\pi n/4}$$

X(7) = x(0) +x(1) $e^{-j7\pi/4}$+x(2) $e^{-j7\pi/2}$+x(3) $e^{-j21\pi/4}$+x(4) $e^{-j7\pi}$+x(5) $e^{-j35\pi/4}$+x(6) $e^{-j21\pi/2}$ +x(7) $e^{-j49\pi/4}$

= 1+0.707+j0.707+j-0.707+j0.707-1-0.707-j0.707

= -0.707+j1.707

X(K)={6,0.707-j1.707,1-j,0.707+j0.293,0,0.707-j0.293,1+j,-0.707+j1.707}

****

H.W: 1. Find the DFT of the Sequence $x(n) = \{1,1,0,0\}$; **Ans:** $X(k) = \{2, 1-j, 0, 1+j\}$

2. Determine the 4-point DFT of the sequence $x(n) = \{0,2,4,6\}$**;** **Ans:** $X(k) = \{12, -4+j4, -4, -4-j4\}$

Challenge 3: Find IDFT of the sequence X (K) = (5,0,1-j,0,1,0,1+j,0)

Solution:

We have

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) e^{j2\pi kn/N} n = 0,1,....,N-1$$

For N=8

$$x(n) = \frac{1}{8}\sum_{k=0}^{N-2} X(k)\, e^{j\pi kn/4} n = 0,1,....7$$

For n=0; $x(0)=\sum_{k=0}^{7} X(k) = \frac{1}{8}$ [5+0+1-j+0+1+j+0]=1

For n=1; $x(1)=\frac{1}{8}\sum_{k=0}^{7} X(k)\, e^{j\pi k/4} = \frac{1}{8}$ [5+-(1-j)(j)+1(-1)+(1+j)(-j)]= 6/8= 0.75

For n=2; $x(2)=\frac{1}{8}\sum_{k=0}^{7} X(k)\, e^{j\pi k/2} = \frac{1}{8}$ [5+(1-j)(-1)+1(1)+(1+j)(-1)]=4/8 =0.5

For n=3; $x(3)=\frac{1}{8}\sum_{k=0}^{7} X(k)\, e^{j3\pi k/4} = \frac{1}{8}$ [5+(1-j)(-j)+1(-1)+ (1+j)(j)]= 2/8 = 0.25

For n=4; $x(4)=\frac{1}{8}\sum_{k=0}^{7} X(k)\, e^{j5\pi k/4} = \frac{1}{8}$ [5+(1-j)(1)+1(1)+(1+j)(1)]=8/8=1

For n=5; $x(5)= \frac{1}{8}\sum_{k=0}^{7} X(k)\, e^{j5\pi k/4} = \frac{1}{8}$ [5+(1-j)(j)+(1)(-1)+(1+j)(-j)]=6/8=0.75

For n=6; $x(6)=\frac{1}{8}\sum_{k=0}^{7} X(k)\, e^{j3\pi k/2} = \frac{1}{8}$ [5+(1-j)(-1)+1(1)+(1+j)(-j)]=4/8=0.5

For n=7;$x(7)=\frac{1}{8}\sum_{k=0}^{7} e^{j7\pi k/4} = \frac{1}{8}$ [5+(1-j)(-j)+1(1)+(1+j)(j)]=2/8=0.25

x(n) = {1,0.75,0.5,0.25,1,0.75,0.5,0.25}

****

Challenge 4: Find IDFT of the sequence X (K) = {1,0,1,0}

Solution:

We have

$$y(n) = \frac{1}{N}\sum_{k-0}^{N-1} Y(k) e^{\frac{j2\pi kn}{N}}, \quad n = 0,1....N-1$$

For N=4

$$y(n)=\frac{1}{4}\sum_{k-0}^{3}Y(k)e^{\frac{j2\pi kn}{4}}, \quad n=0,1,2,3$$

For n=0

$$y(0)=\frac{1}{4}\sum_{k=0}^{3}Y(k)$$

$$=\frac{1}{4}\left[y(0)+y(1)+y(2)+y(3)\right]$$

$$=\frac{1}{4}\left[1+0+1+0\right]$$

$$=0.5$$

For n=1

$$y(1)=\frac{1}{N}\sum_{k=0}^{3}y(k)e^{\frac{j\pi k}{2}}$$

$$=\frac{1}{4}\left[y(0)+y(1)e^{\frac{j\pi}{2}}+y(2)e^{j\pi}+y(3)e^{\frac{j3\pi}{2}}\right]$$

$$=\frac{1}{4}\left[1+0+\cos\pi+j\sin\pi\right]$$

$$=\frac{1}{4}\left[1+0-1+0\right]=0$$

For n=2

$$y(2)=\frac{1}{N}\sum_{k=0}^{3}y(k)e^{\frac{j2\pi k}{2}}$$

$$=\frac{1}{4}\left[y(0)+y(1)e^{j\pi}+y(2)e^{j2\pi}+y(3)e^{j3\pi}\right]$$

$$=\frac{1}{4}\left[1+0+\cos 2\pi+j\sin 2\pi\right]$$

$$=\frac{1}{4}\left[1+0+1+0\right]=0.5$$

For n=3

$$y(3)=\frac{1}{4}\left[y(0)+y(1)e^{\frac{j3\pi}{2}}+y(2)e^{j3\pi}+y(3)e^{\frac{j9\pi}{2}}\right]$$

$$=\frac{1}{4}\left[1+0+\cos 3\pi+j\sin 3\pi\right]$$

$$=\frac{1}{4}\left[1+0-1+0\right]=0$$

$$y(k)=\{0.5,0,0.5,0\}$$

****

H.W: 1. Find the IDFT of the sequence $X(k)=\{1,-2-j-0,-2+j\}$ **with N=4.**

Ans: $x(n)=[-0.75, 0.75, 1.25, -0.25]$

2. Find the 4-point IDFT of the sequence $X(k)=\{2,1-j,0,1+j\}$**;** **Ans:** $x(n)=\{1,1,0,0\}$

****

Challenge 5: Perform the circular convolution of the following sequences

$x_1(n)=\{1,1,2,1\}$ *and* $x_2(n)=\{1,2,3,4\}$ *using DFT and IDFT method.*

Solution:

Given: the circular convolution of the following sequences

$x_1(n) = \{1,1,2,1\}\ and\ x_2(n) = \{1,2,3,4\}$

To find DFT of *x1(k):*

$$X_1(k) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

To find DFT of *x2(k)*:

$$X_2(k) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 10 \\ -2+j2 \\ -2 \\ -2-j2 \end{bmatrix}$$

$$X_3(k) = X_1(k) * X_2(k) = \{5,-1,1,-1\} * \{10,-2+j2,-2,-2-j2\} = \{50,2-j2,-2,2+j2\}$$

To find IDFT of *X3(n)*:

$$X_3(n) = \frac{1}{N} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 10 \\ 2-j2 \\ -2 \\ 2+j2 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 10 \\ 2-j2 \\ -2 \\ 2+j2 \end{bmatrix}$$

$$= \begin{bmatrix} 13 \\ 14 \\ 11 \\ 12 \end{bmatrix}$$

****

Properties of DFT:

1. State and prove any four properties of DFT. May/June-2012] [Nov/Dec-2012] [May/June-2013][Nov/Dec-2013] [Nov/Dec-2014]

1. Periodicity property
2. Linearity property
3. Symmetry property
4. Multiplication of two DFT and convolution
5. Time reversal sequence
6. Parseval's theorem

<u>Periodicity property</u>

If X(k) is N-point DFT of a finite duration sequence x(n)

Then $x(n+N) = x(n) for\ all\ n$

$X(k+N) = X(k) for\ all\ k$

<u>Linearity</u>

If two finite duration sequences x1(n) and x2(n) are linearly combined as

$$x_3(n) = ax_1(n) + bx_2(n)$$

Then the DFT of x3(n) is

$$X_3(k) = aX_1(k) + bX_2(k)$$

Proof:

In summary if, $DFT[x_1(n)] = x_1(k)\ and$

$DFT[x_2(n)] = x_2(k)$

Then

$$DFT[ax_1(n) + bx_2(n)] = aX_1(k) + bX_2(k)$$

<u>Time Reversal of the sequence</u>

$$x\big((-n)\big)N = x(N-n) \quad 0 \le n \le N-1$$

$$DFT\big[x\big((-n)\big)N\big] = DFT[x(N-n)] = x\big((-k)\big)N = X(N-K)$$

Proof:

$$DFT[x(N-n) = \sum_{n=0}^{N-1} x(\text{n})\,\text{e}^{-j2\pi kn/N}$$

Changing the index from n to m = N-n, we get

$$DFT[x(N-m) = \sum_{n=0}^{N-1} x(m)\,\text{e}^{-j2\pi k(n-m)/N}$$

$$= X(N-k)$$

<u>Complex conjugate property</u>

If $DFT[x(n)] = X(k)$ then

$DFT[x^*(n)] = X^*(N-k) = X^*((-k))N$

Proof:

$$DFT[x^*(n)] = \sum_{n=0}^{N-1} x^*(n)e^{-j2\pi kn/N}$$

$$= [\ \textstyle\sum_{n=0}^{N-1} x(n)e^{j2\pi kn/N}]^*$$

$$= [\ \textstyle\sum_{n=0}^{N-1} x(n)e^{-\frac{j2\pi n(N-k)}{N}}]^* \qquad = X^*(N-k)$$

$$DFT[x^*(N-n)] = X^*(k)$$

<u>Multiplication of Two sequences</u>

If $DFT[x_1(n)] = X_1(k)$ and

$DFT[x_2(n)] = X_2(k)$ then

$$DFT[x_1(n)x_2(n)] = \frac{1}{N}[X_1(k)\ \odot X_2(k)]$$

<u>Parseval's Theorem:</u>

$DFT[x(n)] = X(k)$

and $DFT[y(n)] = Y(k)$

Then

$$\sum_{n=0}^{N-1} x(n)y^*(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)Y^*(k)$$

<u>FFT-Fast Fourier Transform</u>

- DFT is given by

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-\frac{j2\pi kn}{N}}; \quad k = 0,1,2........N-1.$$

- The direct computation of the DFT requires $2N^2$ evaluations of trigonometric functions, $4N^2$ real multiplications 4N (N-1) real additions.

By using the twiddle factor,

$$W_N^{nk} = e^{-\frac{j2\pi nk}{N}}$$

Where, W_N is the "N^{th}" root of unity.

Properties of twiddle Factor:

$$(i)\ W_N^{k+\frac{N}{2}} = -W_N^k$$

$$(ii)\ W_N^{k+N} = W_N^k$$

There are two types of Radix-2 FFT

1. Decimation in Time FFT
2. Decimation in Frequency FFT

Decimation in Time FFT

2. **Draw and explain the basic butterfly diagram or flow graph of DIT radix-2 FFT.[Nov/Dec-2009]**

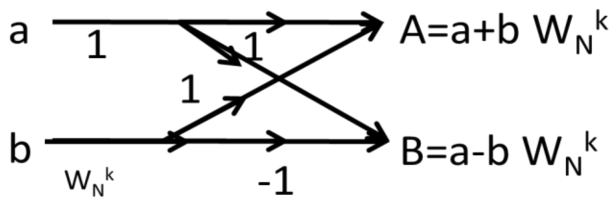
1.In each computation two complex numbers" and "b" are considered.

2. The complex number "b" is multiplied by a phase factor "W_N^{kn}"

3. The product " bW_N^{kn}" is added to complex number "a" to form new complex number "A"

4. The product " bW_N^{kn}" is subtracted from comblex number "a" to form new complex number "B".

The above basic computation can be expressed by a signal flow graph

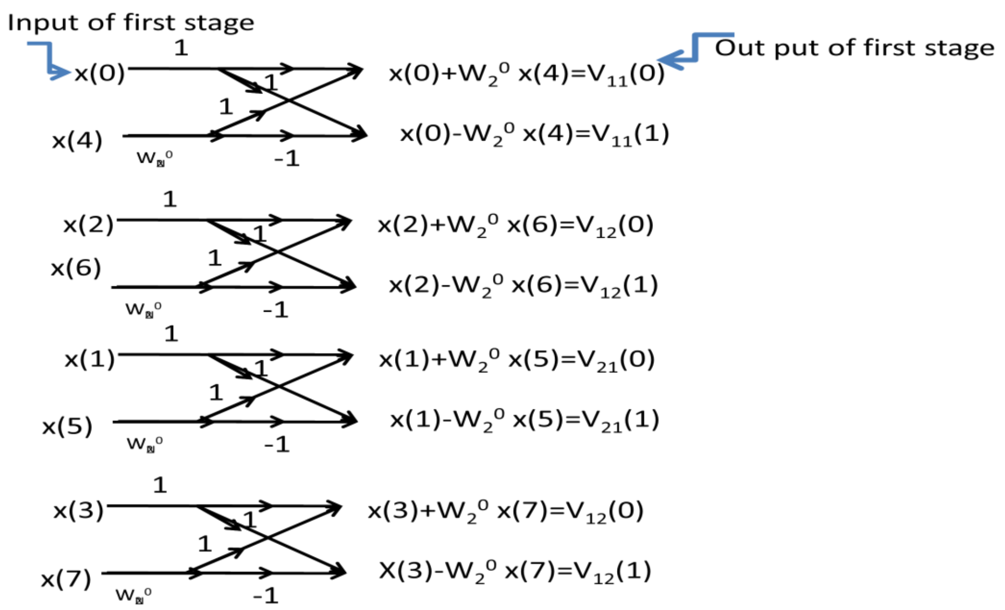


The signal flow graph is also called butterfly diagram since it resembles a butterfly. In radix-2 FFT, Z/2 butterflies per stage are required to represent the computational process. The butterfly diagram used to compute the 8point DFT via radix-2 DIT FFT.

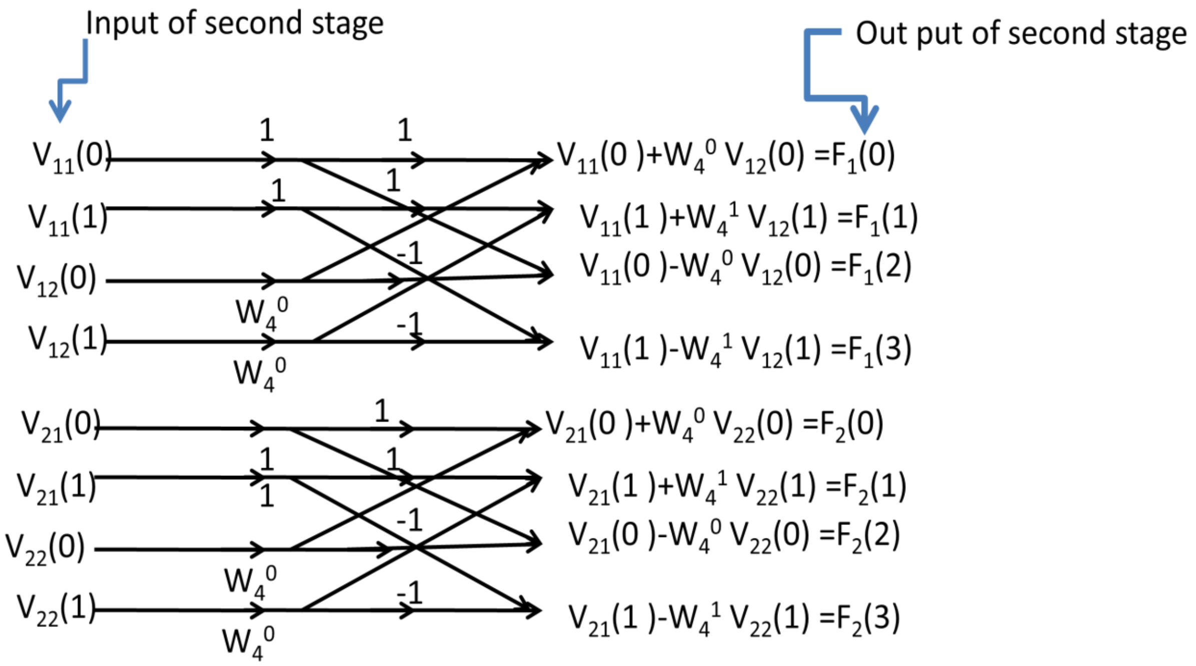
The sequence x (n) is arranged in bit reversed order and then decimated into two sample sequences.

x(0)	x(2)	x(1)	x(3)
x(4)	x(6)	x(5)	x(7)

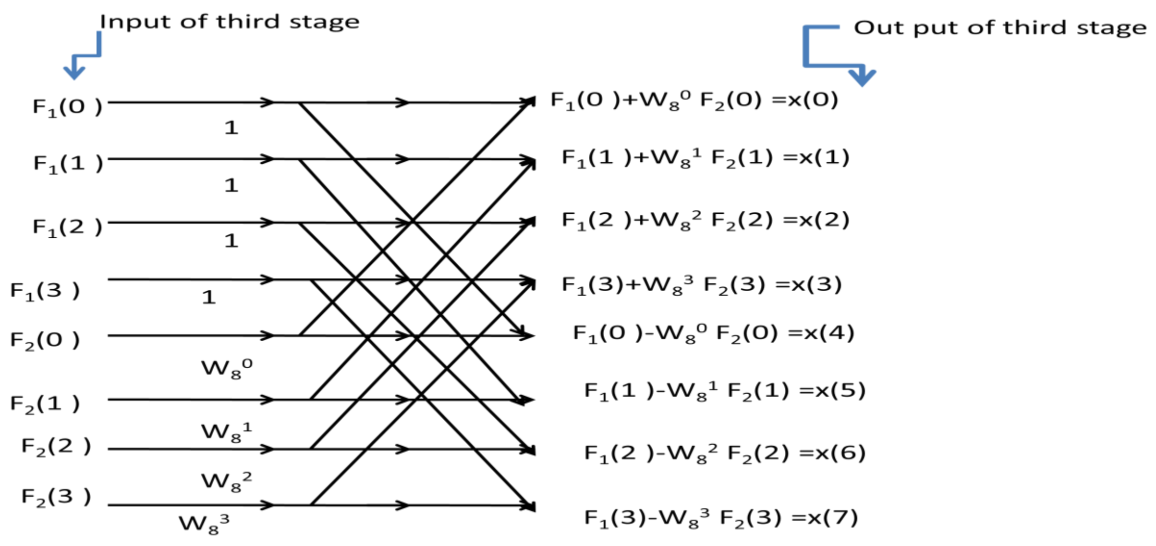
Fist stage of flow graph for 8-point DFT via radix-2 DIT FFT



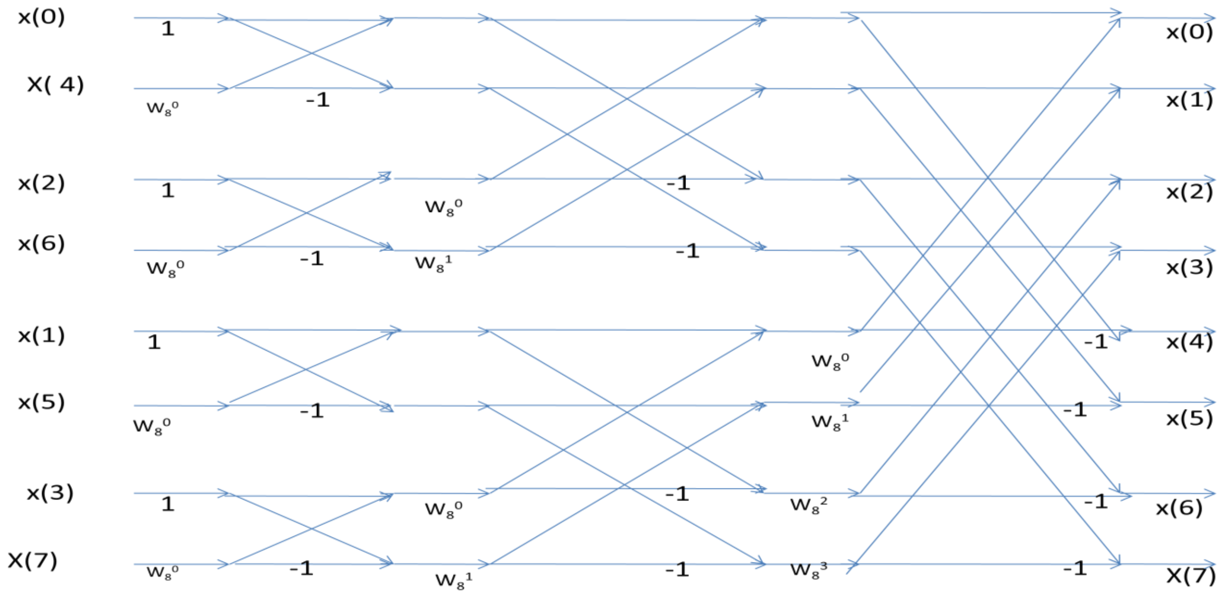
Second stage of flow graph for 8-point DFT via radix-2 DIT FFT



Third stage of flow graph for 8-point DFT via radix-2 DIT FFT



Combined stage for computation:

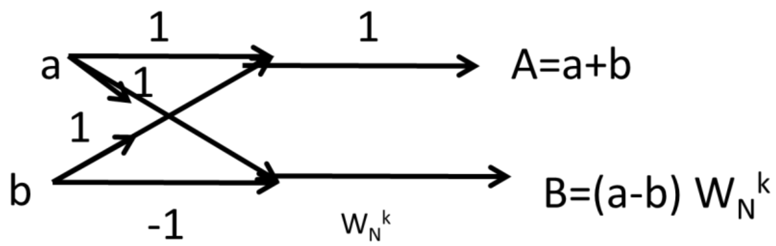


Decimation in Frequency FFT

3. Draw and explain the basic butterfly diagram or flow graph of DIF radix-2 FFT.[Nov/Dec-2012]

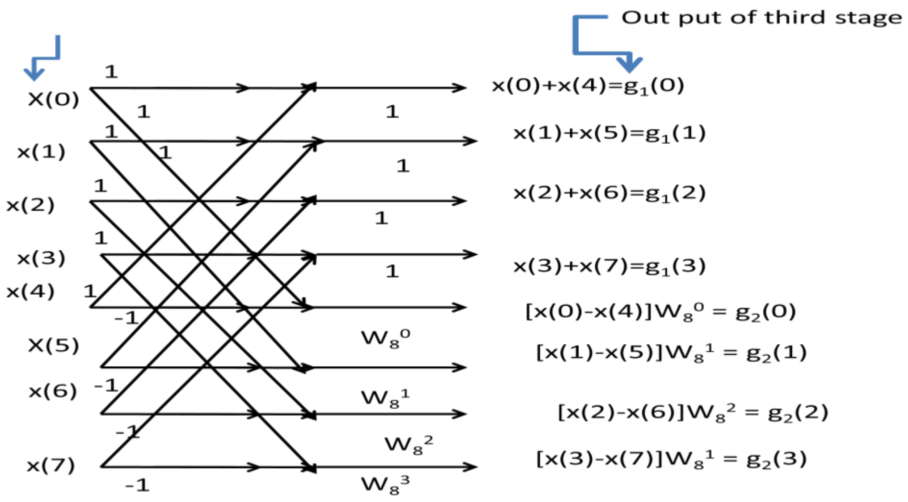
1. In each computation two complex numbers "a" and "b" are considered.
2. The sum of the two complex number a&b are considered.
3. The subtract complex number "b" from "a" to get the term "a-b" . The difference term "a-b" is multiplied with the phase factor or twiddle factor "W_N^k" to form a new complex number"B".

The above basic computation can be expressed by a signal flow graph

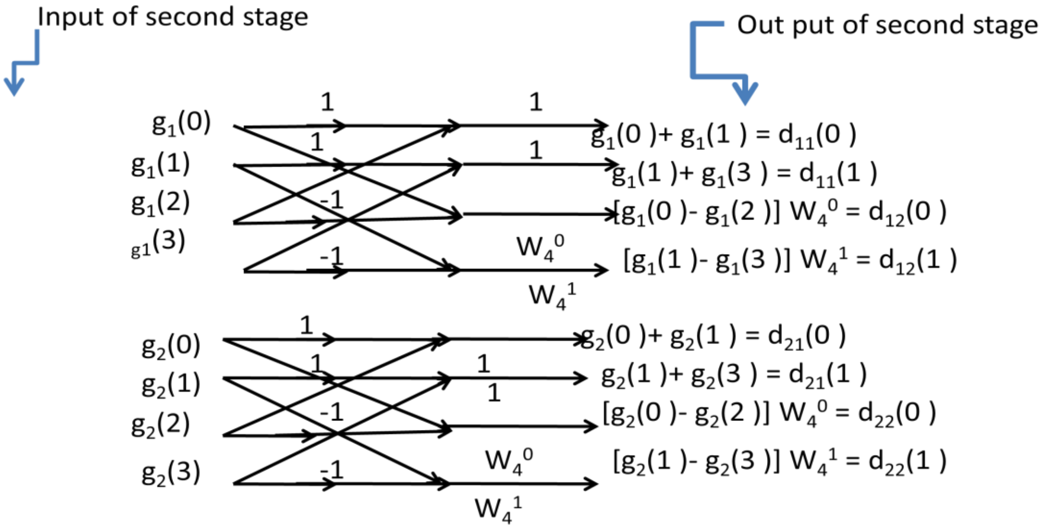


The signal flow graph is also called butterfly diagram since it resembles a butterfly. In radix-2 FFT, Z/2 butterflies per stage are required to represent the computational process. The butterfly diagram used to compute the 8point DFT via radix-2 DIF FFT

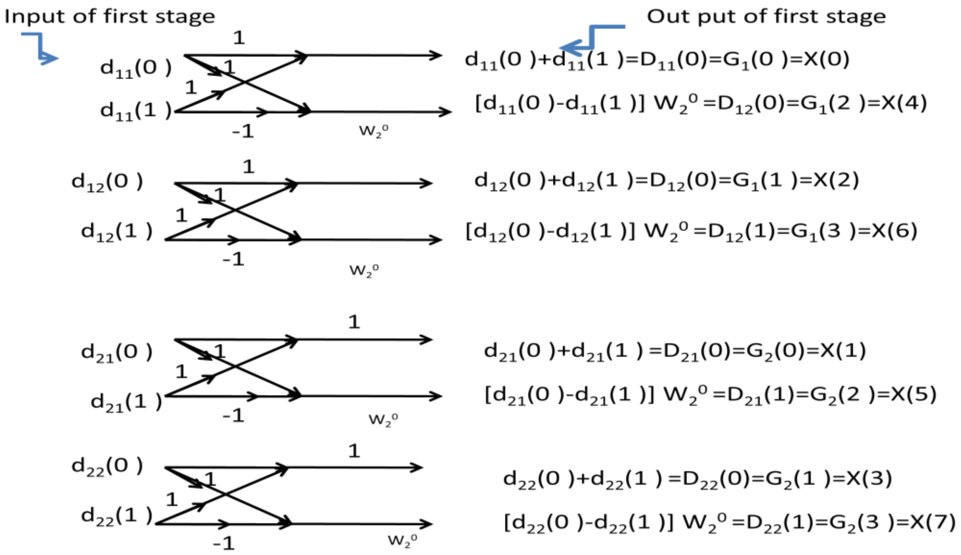
Flow graph for Fist Stage of Computation:



Flow graph or butterfly diagram for second stage of computation:



Flow graph for Third Stage of computation:



Challenge 6: Find the DFT of the sequence x[n] = {1, 2, 3, 4, 4, 3, 2, 1} using radix-2 decimation in time FFT algorithm. [Nov/Dec-2009] [April/May-2011][Nov/Dec-2012] [Nov/Dec-2012][May/June-2013] [Nov/Dec2013][April/May-2014][Nov/Dec-2014][May/June-2015] [May/June-2016].

Solution:

$Given = \{1, 2, 3, 4, 4, 3, 2, 1\}$

$We\,know\,that\, W_N^k = e^{-j\left(\frac{2\pi}{N}\right)k}$

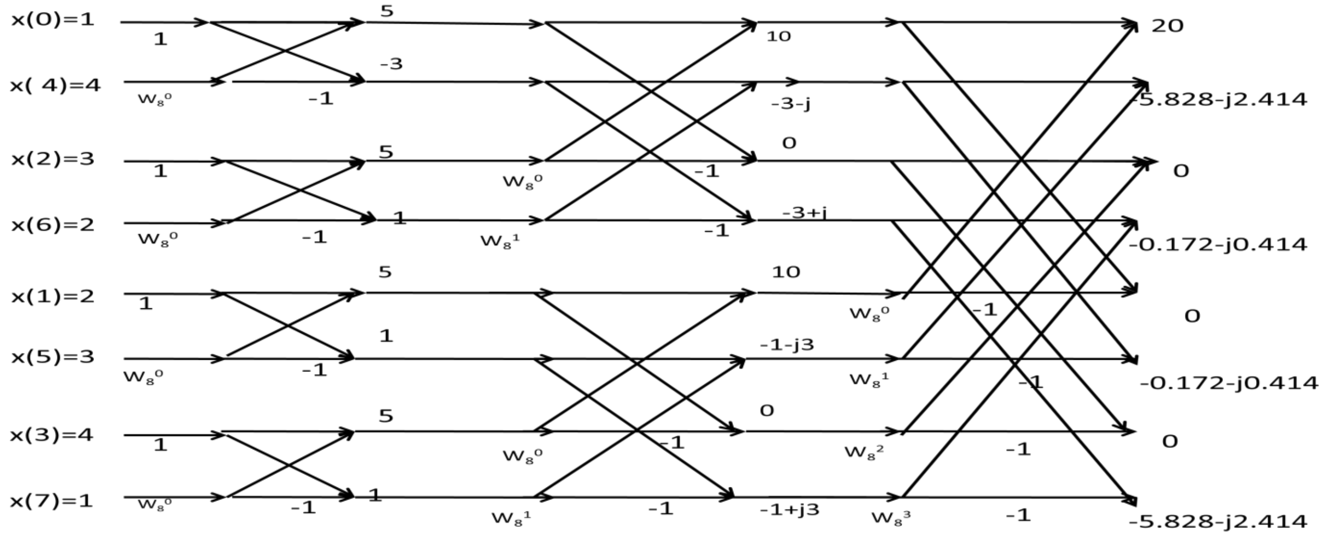
$Given\;\; N = 8$

$Hence, W_8^1 = e^{-j\left(\frac{2\pi}{8}\right)0} = 1$

$= W_8^1 = e^{-j\left(\frac{2\pi}{8}\right)1} = \cos\frac{\pi}{4} - j\sin\frac{\pi}{4} = 0.707 - j0.707$

$= W_8^2 = e^{-j\left(\frac{2\pi}{8}\right)2} = \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} = -j$

$= W_8^3 = e^{-j\left(\frac{2\pi}{8}\right)3} = \cos\frac{3\pi}{4} - j\sin\frac{3\pi}{4} = -0.707 - j0.707$



$X(k) = \{20, -5.828 - j2.414, 0, -0.172 - j0.414, 0, -0.172 + j0.414, 0 - 5.828 + j2.414\}$

X(k)={2,0.5-j1.207,0,0.5-j0.207,0,0.5-j0.207,0,0.5-j1.207}

****

H.W: 1.Find the 8-point DFT of the sequence $x(n) = \{2,2,2,2,1,1,1,1\}$ **using DIT-FFT algorithm.**

Ans: $X(k) = \{12, 1 - j2.414, 0, 1 - j0.4142, 0, 1 + 0.4142, 0, 1 + j2.414\}$

2. Find the 8-point DFT of the sequence $x(n) = \begin{cases} 1 \; for -3 \le n \le 3 \\ 0 \; otherwise \end{cases}$ **using DIT-FFT algorithm.**

Ans: $X(k) = \{7, 1, -1, 1, -1, 1, -1, 1\}$

3. Find the 8-point DFT of the sequence $x(n) = \begin{cases} 1 \; for \; 0 \le n \le 6 \\ 0 \; otherwise \end{cases}$ **using DIT-FFT algorithm.**

Ans: $X(k) = \{7, -0.707 - j0.707, -j, 0.707 - j0.707, 1, 0.707 + j0.707, j, -0.707 + j0.707\}$

4. Given $x(n) = 2^n \; and \; N = 8$ **, find *X(k)* using DIT-FFT algorithm.**

Ans: $X(k) = \{255, 48.63 + j166, -51 + j102, -78.63 + j46.05, -85, -78.63 - j46.05, -51 - j102, 48.63 - j166\}$

5. Compute the 8-point DFT of the sequence x (n) = (0.5, 0.5, 0.5, 0.5, 0, 0, 0, 0) using the radix-2 DIT algorithm.

X(k)={2,0.5-j1.207,0,0.5-j0.207,0,0.5-j0.207,0,0.5-j1.207}

6. Find the 8-point DFT of the sequence $x(n) = \{1, -1, -1, -1, 1, 1, 1, -1\}$ **using DIT-FFT algorithm.**

$x(k) = \{0, -1.414 + j3.414, 2 - 2j, 1.44 - j0.586, 4, 1.414 + 0.586, 2 + j2, -1.414 - j3.414\}$

7. Given $x(n) = n + 1 \; and \; N = 8$ **, find *X(k)* using DIT-FFT algorithm.**

Ans: $X(k) = \{36, -4 + j9.656, -4 + j4, -4 + j1.656, -4, -4 - 1.656, -4 - 4j, -4 - j9.656\}$

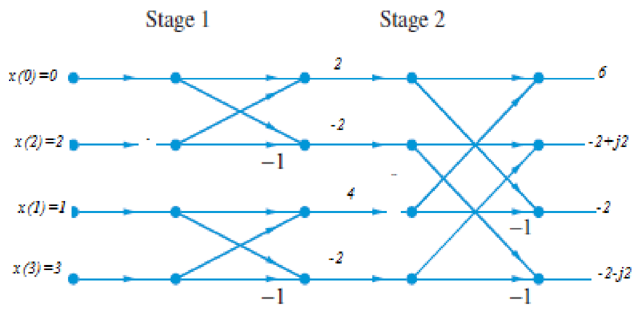
Challenge 7: Compute 4-point DFT of a sequence $x(n) = \{0,1,2,3\}$ using DIT algorithm.

Solution:

Given: The DFT of the sequence $x(n) = \{0,1,2,3\}$

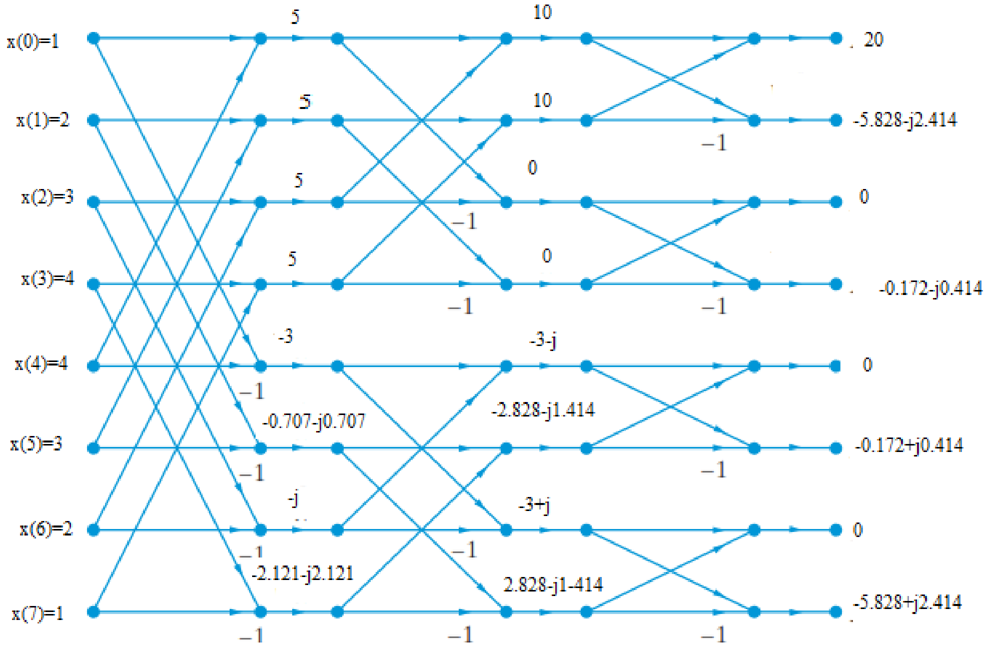
The twiddle factors are,

$$W_4^0 = 1;\ W_4^1 = -j$$



$X(k) = \{6, -2 + j2, -2, -2 - j2\}$

Problem 8: Find the 8-point DFT of given sequence $x(n) = \{1,2,3,4,4,3,2,1\}$ using DIF-FFT algorithm. [May/June 2015] [Nov/Dec-2014]



$X(k) = \{20, -5.828 - j2.414, 0, -0.172 - j0.414, 0, -0.172 + j0.414, 0 -5.828 + j2.414\}$

****

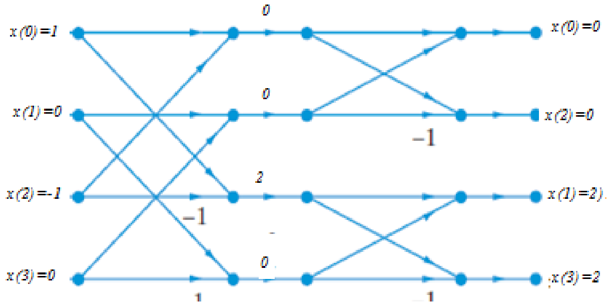
Challenge 8: Compute the DFT of the sequence $x(n) = \cos\frac{n\pi}{2}$,where N=4 using DIF-FFT algorithm.

Solution:

Given: Given: The DFT of the sequence $x(n) = \cos\frac{n\pi}{2}$

The twiddle factors are,

$$W_4^0 = 1;\ W_4^1 = -j$$



$X(k) = \{0,2,0,2\}$

****

H.W:1. Compute the FFT for the sequence $x(n) = \{1,0,0,0,0,0,0,0\}$ **Ans:** $X(k) = \{1,1,1,1,1,1,1,1\}$

2. Find the 8-point DFT of the given sequence $x(n) = \{0,1,2,3,4,5,6,7\}$ **using DIF, radix-2 FFT algorithm.**

Ans: $X(k) = \{28, -4 + j9.656, -4 + j4, -4 + j1.656, -4, -4 - j1.656, -4 - j4, -4 - j9.656\}$

3. Find the 8-point DFT of the sequence $x(n) = \{1,2,2,1,1,2,2,1\}$ **using DIF-FFT algorithm.**

Ans: $X(k) = \{12, 0, 2 - j2, 0, 0, 0, -2 + j2, 0\}$

4.Find the IDFT of the sequence $X(k) = \{4, 1 - j2.414, 0, 1 - j0.414, 0, 1 + j0.414, 0, 1 + 2.414\}$ **using DIF algorithm.** **Ans:** $x(n) = \{1,1,1,1,0,0,0,0\}$

5. Find the 8-point DFT of given sequence $x(n) = \{2,2,2,2,1,1,1,1\}$ **using DIF-FFT algorithm. [May/June-2015] [Nov/Dec-2014]**

X(k)= {12, 1-j2.414, 0, 1-j0.4142, 0, 1+j0.4142, 0, 1+j2.414}

6. Given $x(n) = n + 1$ *and* $N = 8$ **, find *X(k)* using DIF-FFT algorithm.**

Ans: $X(k) = \{36, -4 + j9.656, -4 + j4, -4 + j1.656, -4, -4 - 1.656, -4 - 4j, -4 - j9.656\}$

****

Filtering Methods based on DFT: **Overlap-save method**

Filtering of a long sequence → **Overlap-add method**

4: Explain overlap add method for linear FIR filtering of a long sequence.

- Let the length of the sequence be L_S and the length of the impulse response is M.
- The sequence is divided into blocks of data size having length L and M-1 zeros are appended to it to make the data size of L+M-1.
- Thus the data blocks may be represented as

$x_1(n)$ = (x(0),x(1,…..,x(L-1), 0,0,……….) → M-1 zeros appended

$x_2(n)$= (x(L),x(L+1),…….,x(2L-1), 0,0,……)

$x_3(n)$ = (x(2L),x(2L+1),…….x(3L-1), 0,0……..) → M-1 zeros appended

- Now L-1 zeros are added to the impulse response h (n) and N-point circular convolution is performed.
- Since each data block is terminated with M-1 zeros, the last M-1 points from each output block must be overlapped and added to the first M-1 points of the succeeding block. Hence this method is called overlap-add method.
- Let the output blocks are of the form

$y_1(n)$ = ($y_1(0)$, $y_1(1)$,……… y_1(L-1,), $y_1(L)$,……………. $y_1(N-1)$)
$y_2(n)$ = ($y_2(0)$, $y_2(1)$,……… y_2(L-1,), $y_2(L)$,……………. $y_2(N-1)$)
$y_3(n)$ = ($y_3(0)$, $y_3(1)$,……… y_3(L-1,) , $y_3(L)$,……………. $y_3(N-1)$)

The output séquence is

y(n)=($y_1(0)$, $y_1(1)$,……… y_1(L-1,) ,$y_1(L)$+ $y_2(0)$, $y_2(1)$,…………., $y_1(N-1)$+ $y_2(M-2)$,$y_2(M)$,……….., $y_2(L)$+ $y_3(0)$, $y_2(L+1)$+ $y_3(1)$,…………………., $y_3(N-1)$)

OVERLAP SAVE METHOD OF LINEAR FILTERING

Step 1 : In this méthod L samples of the current segment and M-1 samples of the previous segment forms the input data block. Thus data block will be

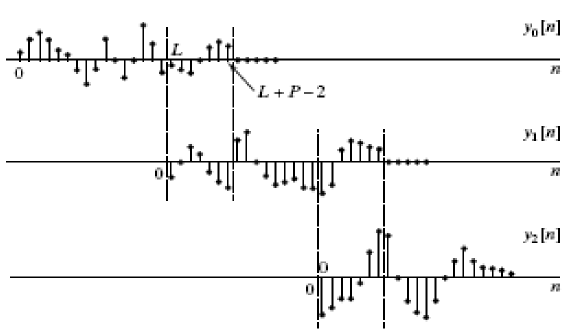
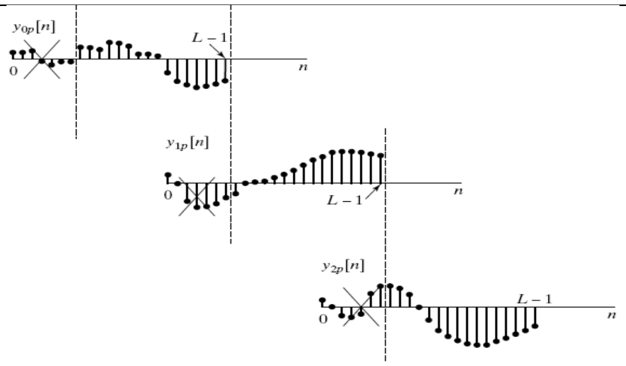
X1(n) ={0,0,0,0,0,………………… ,x(0),x(1),…………….x(L-1)}
X2(n) ={x(L-M+1), …………….x(L-1),x(L),x(L+1),,,,,,,,,,,,x(2L-1)}
X3(n) ={x(2L-M+1), …………….x(2L-1),x(2L),x(2L+2),,,,,,,,,,,,x(3L-1)}

Step2 : Unit sample response h(n) contains M samples hence its length is made N by padding zeros. Thus h(n) also contains N samples.
h(n)={ h(0), h(1), …………….h(M-1), 0,0,0,……………………(L-1 zeros)}

Step3 : The N point DFT of h(n) is H(k) & DFT of mth data block be xm(K) then corresponding DFT of output be $Y_m(k)$
$Y_m(k)$= H(k) xm(K)

Step 4 : The sequence ym(n) Can be obtained by taking N point IDFT of Y`m(k). Initial (M-1) samples in the corresponding data block must be discarded. The last L samples are the correct output samples. Such blocks are filtered one after another to get the final output.

5. Summarize the difference between overlap save and add method.

Overlap add Method	**Overlap save Method**
The overlap-add procedure cuts the signal up into equal length segments with no overlap.	The overlap-save procedure cuts the signal up into equal length segments with some overlap
Then it zero-pads the segments and takes the DFT of the segments. Part of the convolution result corresponds to the circular convolution	Then it takes the DFT of the segments and saves the parts of the convolution that correspond to the circular convolution
Results in the aliasing that occurs in circular convolution.	No lost information in throwing away parts of the linear convolution.
$x_r[n] = \begin{cases} x[n+rL] & 0 \le n \le L-1 \\ 0 & otherwise \end{cases}$	$x_r[n] = x[n + r(L-P+1) - P + 1], \quad 0 \le n \le L-1$
$y[n] = x[n] * h[n] = \sum_{m=-\infty}^{\infty} y_r[n - rL]$	$y[n] = \sum_{r=0}^{\infty} y_r[n - r(L-P+1) + P - 1]$
 Fir filtering by using the overlapping-add method	 Result of circularly convolving each section with h[n]. The portions of each filter section to be discarded in forming the linear convolution are indicated

Challenge 9: Determine the output response y(n) if h(n)=(1,1,1), x(n)=(1,2,3,1) by using linear convolution, circular convolution & circular convolution with zero padding. (16)

Solution: Linear convolution:

x (n) = {1,2,3,1}, h(n)= {1,1,1}

	1	1	1	0 h(n)
1	1	1	1	0
2	2	2	2	0
3	3	3	3	0
1	1	1	1	0
x(n)				

$y(n) = \{\underset{\uparrow}{1}, 3, 6, 6, 4, 1\}$

Number of samples in linear convolution is L+M-1 = 4+3-1 = 6

Circular convolution:

x(n) = {1,2,3,1} ; h(n) = {1,1,1,0}

Using matrix approach

$$\begin{vmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 \\ 2 \\ 3 \\ 4 \end{vmatrix} = \begin{vmatrix} 5 \\ 4 \\ 6 \\ 6 \end{vmatrix}$$

y(n) = x(n) ©h(n) ={5,4,6,6}

By comparing circular convolution output with that of linear convolution we find that the first 2 points (M-1) are aliased ie. Last two data points are added to first two data points in linear convolution are added to first two data points as shown below. 1+4=5 & 3+1=4

Circular convolution with zero padding:

Add (M-1) zeros with x(n) and (L-1) zeros with h(n); x(n) = {1,2,3,1,0,0} h(n) = {1,1,1,0,0,0}

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 \\ 2 \\ 3 \\ 1 \\ 0 \\ 0 \end{vmatrix} = \begin{vmatrix} 1 \\ 3 \\ 6 \\ 6 \\ 4 \\ 1 \end{vmatrix}$$

y(n) = (1,3,6,6,4,1)

****

Challenge 10: Find the output y(n) of a filter whose impulse response is h(n)=(1,1,1) and input signal x(n)=(3,-1,0,1,3,2,0,1,2,1) using over lab save and add method. [May/June-2016]

Solution:

Overlap save method:

$x_1(n)$= {0,0, 3,-1,0}

M-1=2 zeros   L=3 data points

$x_2(n)$= {-1,0,1,3,2}

2 data from previous   3 new data

$x_3(n)$= {3,2,0,1,2} and $x_4(n)$= {1,2,1,0,0}

Given h(n) = {1,1,1}

Increase the length by adding zeros(L+M-1=5)

i.e. h(n) = {1,1,1,0,0}

$y_1(n)$ = $x_1(n)$©h(n) = {-1,0,3,2,2}

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 3 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 3 \\ 2 \\ 2 \end{bmatrix}$$

$y_2(n)$ = $x_2(n)$ © h(n) = {4,1,0,4,6}

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 4 \\ 6 \end{bmatrix}$$

$y_3(n) = x_3(n)$ © $h(n) = \{6,7,5,3,3\}$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \\ 5 \\ 3 \\ 3 \end{bmatrix}$$

$y_4(1) = x_4(n)$ © $h(n) = \{1,3,4,3,1\}$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \\ 3 \\ 1 \end{bmatrix}$$

[**-1,0**,3,2,2]
[**4,1,**0,4,6]
[**6,7,**5,3,3]
[**1,3,**4,3,1]

(by discarding the selected)

Y(n) = {3,2,2,0,4,6,5,3,3,4,3,1}

Overlap add method:

Length: L+M-1=5; N_2-1=3-1=2

Therefore,

$x_1(n) = \{3,-1,0\}$
$x_2(n) = \{1,3,2\}$
$x_3(n) = \{0,1,2\}$
$x_4(n) = \{1,0,0\}$

$y_1(n) = x_1(n).h(n) = \{3,2,2,-1,0\}$

	3	-1	0
1	3	-1	0
1	3	-1	0
1	3	-1	0

Asdy(n)={3,2,2,-1,0}

Similarly , $y_2(n) = x_2(n)$ © $h(n) = \{1,4,6,5,2\}$
$y_3(n) = x_3(n)$ © $h(n) = \{0,1,3,3,2\}$
$y_4(1) = x_4(n)$ © $h(n) = \{1,1,1,0,0\}$

[3,2,2,-1,0]
[1,4,6,5,2] by adding
[0,1,3,3,2]

[<u>1,1,</u>1,0,0]

y(n) = {3,2,2,0,4,6,5,3,3,4,3,1}

****

H.W:

1. Using linear convolution find y(n)=x(n)*h(n) for the sequences x(n)=(1,2,-1,2,3,-2,-3,-1,1,1,2,-1) and h(n)=(1,2).Find the result by solving overlap-save and overlap add method.

Solution:

Linear convolution:

	1	2	-1	2	3	-2	-3	-1	1	1	2	-1
1	1	2	-1	2	3	-2	-3	-1	1	1	2	-1
2	2	4	-2	4	6	-4	-6	-2	2	2	4	-2

y(n)= (1,4,3,0,7,4,-7,-7,-1,3,4,3,-2)

2. **Perform linear convolution of finite duration sequence** $h(n) = \{1,1,2,1\}$ **and** $x(n) = \{1,-1,1,2,1,0,1,-4,3,2,1,0,1,1\}$ **by overlap-add and overlap-save methods.**

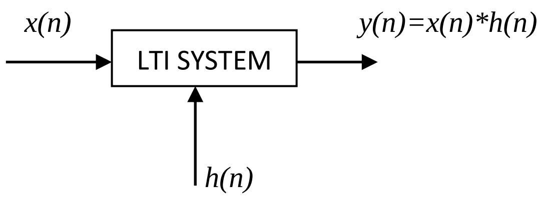
Ans: $y(n) = \{1,0,2,2,4,6,5,-2,1,-2,5,8,5,3,3,3,1\}$

****

<u>**Use of FFT as Linear Filtering:**</u>

6. Write short note on use of FFT as linear Filtering.

Linear filtering is needed in variety of applications. In this approach, input is known as value. If the input *x(n)* is large. Then it is very difficult to process it by using FFT algorithm. So, sectioned convolution concept is used.



The response of LTI system is given by the linear convolution of (*x(n)*input and *h(n))* impulse response. If *x(n) and h(n)* are small in length then, the computation of response is easy. But, one of the sequence *(x(n) (or) h(n))* is larger than other, then following problems may occur.

- ✓ Large amount of memory is required to store the lengthy sequence.
- ✓ Long delay occurs.

In sectioned convolution, large sequence *(x(n))* is divided into (or sectioned into) small sub sequences *((x1(n),x2(n)……..)*, Then, linear convolution of subsequences *(x1(n))*and other sequence *(h(n))* is computed. Finally, the output of all linear convolution (*(x1(n)*h(n)),((x2(n)*h(n)),……)* are combined to form the overall output.

Challenge 11: Find the DFT of the sequence $x(n) = \begin{cases} 1 \; for\, 0 \le n \le 2 \\ 0 \; otherwise \end{cases}$ **for *N=4* and compute the corresponding amplitude and phase spectrum.**

Solution: $X(k) = \sum_{n=0}^{N-1} x(n) e^{\frac{-j2\pi kn}{N}} \quad ; \quad k = 0,1........N-1$

Given: The DFT of the sequence $x(n) = \begin{cases} 1 \; for\, 0 \le n \le 2 \\ 0 \; otherwise \end{cases}$

Here x(0)=1, x(1)=1,x(2)=1,x(3)=0; N=4.

For k=0:

$$X(0)=\sum_{n=0}^{3}x(n)=x(0)+x(1)+x(2)+x(3)$$
$$=3$$
$$Therefore\ |X(0)|=3,\ \angle X(0)=0$$

For k=1:

$$X(1)=\sum_{n=0}^{3}x(n)e^{\frac{-j\pi n}{2}}=x(0)+x(1)e^{\frac{-j\pi}{2}}+x(2)e^{-j\pi}+x(3)e^{\frac{-j3\pi}{2}}$$
$$=1+\cos\frac{\pi}{2}-j\sin\frac{\pi}{2}+\cos\pi-j\sin\pi+0$$
$$=1-j-1=-j$$
$$Therefore\ |X(1)|=1,\ \angle X(1)=\frac{-\pi}{2}$$

For k=2

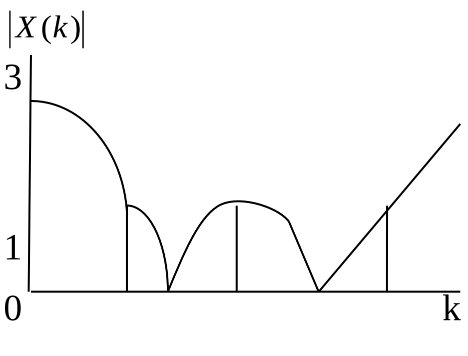
$$X(2)=\sum_{n=0}^{3}x(n)e^{-j\pi n}=x(0)+x(1)e^{-j\pi}+x(2)e^{-j2\pi}+x(3)e^{-j3\pi}$$
$$=1+\cos\pi-j\sin\pi+\cos 2\pi-j\sin 2\pi+0$$
$$=1-1+1=1$$
$$Therefore\ |X(2)|=1,\ \angle X(2)=0$$

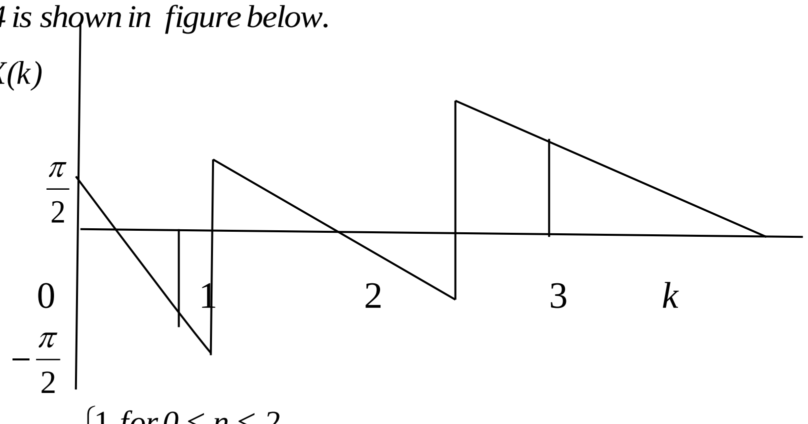
For k=3

$$X(3)=\sum_{n=0}^{3}x(n)e^{\frac{-j3\pi n}{2}}=x(0)+x(1)e^{\frac{-j3\pi}{2}}+x(2)e^{-j3\pi}+x(3)e^{\frac{-j9\pi}{2}}$$
$$=1+\cos\frac{3\pi}{2}-j\sin\frac{3\pi}{2}+\cos 3\pi-j\sin 3\pi+0$$
$$=1+j-1=j$$
$$Therefore\ |X(3)|=1,\ \angle X(3)=\frac{\pi}{2}$$

$$|X(k)|=\{3,1,1,1\} \qquad \angle X(k)=\left\{0,\frac{-\pi}{2},0,\frac{\pi}{2}\right\}$$

The plot for $|X(k)|$ *and* $\angle X(k)$ *for* $N=4$ *is shown in figure below.*





H.W: Find the DFT of the sequence $x(n)=\begin{cases}1 & for\ 0\le n\le 2\\ 0 & otherwise\end{cases}$ **for *N=4* and compute the corresponding amplitude and phase spectrum.**

Ans:
$$|X(k)|=\{3,2.414,1,0.414,1,0.414,1,2.414\}$$
$$\angle X(k)=\left\{0,-\frac{\pi}{4},-\frac{\pi}{2},\frac{\pi}{4},0,-\frac{\pi}{4},\frac{\pi}{2},\frac{\pi}{4}\right\}$$

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UNIT II – INFINITE IMPULSE RESPONSE FILTERS

Characteristics of practical frequency selective filters. characteristics of commonly used analog filters - Butterworth filters, Chebyshev filters. Design of IIR filters from analog filters (LPF, HPF, BPF, BRF) - Approximation of derivatives, Impulse invariance method, Bilinear transformation. Frequency transformation in the analog domain. Structure of IIR filter - direct form I, direct form II, Cascade, parallel realizations.

STRUCTURES FOR IIR SYSTEMS:

IIR Systems are represented in four different ways

1. Direct Form Structures Form I and Form II
2. Cascade Form Structure
3. Parallel Form Structure
4. Lattice and Lattice-Ladder structure.

DIRECT FORM-I :

Challenge: Obtain the direct form-I, direct form-II,Cascade and parallel form realization of the system y(n)=-0.1y(n-1)+0.2y(n-2)+3x(n)+3.6x(n-1)+0.6x(n-2) [April/May-2015]

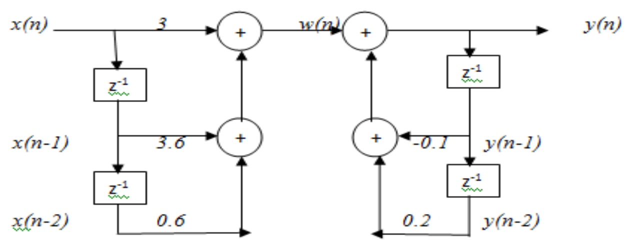
Solution:

Direct Form I:

$$Let\ 3x(n)+3.6x(n-1)+0.6x(n-2)=w(n)$$

$$y(n)=-0.1y(n-1)+0.2y(n-2)+w(n)$$

The direct form I realization is

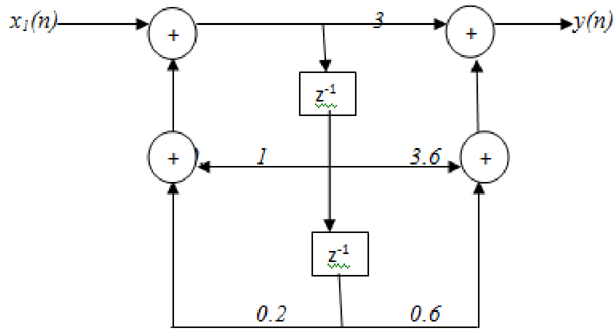


Direct form II:

From the given difference equation we have

$$H(z)=\frac{Y(z)}{X(z)}=\frac{3+3.6z^{-1}+0.6z^{-2}}{1+0.1z^{-1}-0.2z^{-2}}$$

The above system function can be realized in direct form II



Cascade Form:

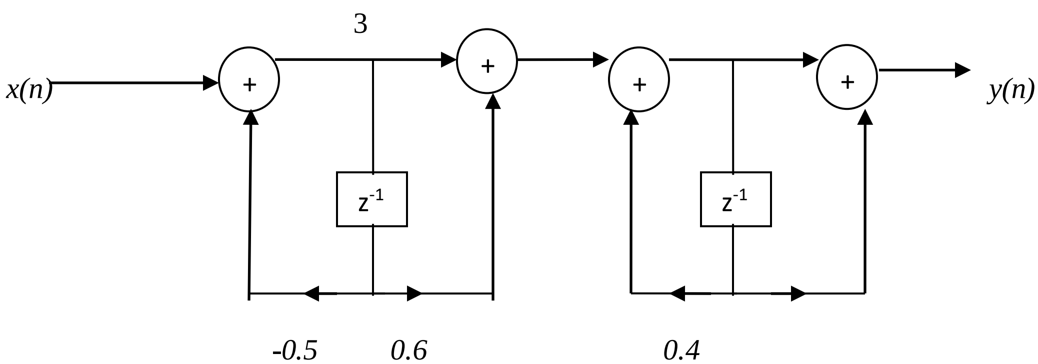
$$\frac{Y(z)}{X(z)} = \frac{3+3.6z^{-1}+0.6z^{-2}}{1+0.1z^{-1}-0.2z^{-2}}$$

$$= \frac{(3+0.6z^{-1})(1+z^{-1})}{(1+0.5z^{-1})(1-0.4z^{-1})}$$

$$H_1(z) = \frac{3+0.6z^{-1}}{1+0.5z^{-1}}$$

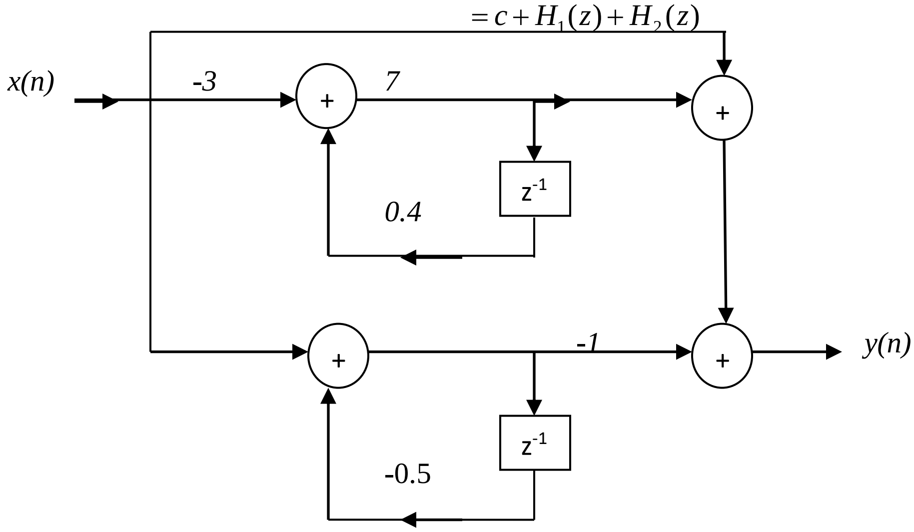
$$H_2(z) = \frac{1+z^{-1}}{1-0.4z^{-1}}$$

Now we realize $H_1(z)$ *and* $H_2(z)$ and cascade both to get realization of H(z)



Parallel form:

$$H(z)=\frac{3+3.6z^{-1}+0.6z^{-2}}{1+0.1z^{-1}-0.2z^{-2}}$$
$$=-3+\frac{7}{1-0.4z^{-1}}-\frac{1}{1+0.5z^{-1}}$$
$$=c+H_1(z)+H_2(z)$$



<u>Direct form I:</u>

H.W: Obtain the direct form-I realization for the system described by the following difference equations.

(i) $y(n)=2y(n-1)+3y(n-2)+x(n)+2x(n-1)+3x(n-2)$

(ii) $y(n)=0.5y(n-1)+0.06y(n-2)+0.3x(n)+0.5x(n-1)$

Obtain the direct form-I realization for the system described by difference equation

$y(n)=0.5y(n-1)-0.25y(n-2)+x(n)+0.4x(n-1)$

<u>Direct form II</u>

H.W: Determine the direct form II realization for the following system

(i) $y(n)+y(n-1)-4y(n-3)=x(n)+3x(n-2)$

(ii) $y(n)=\frac{3}{4}y(n-1)-\frac{1}{8}y(n-2)+x(n)+\frac{1}{2}x(n-1)$ **[May/June-14]**

Determine the direct form II realization for the following system

$y(n)=-0.1y(n-1)+0.72y(n-2)+0.7x(n)-0.252x(n-2)$

<u>CASCADE FORM:</u>

H.W: For the system function $H(z)=\dfrac{1+2z^{-1}+z^{-2}}{1-\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}}$ **obtain cascade structure.**

Realize the system with difference equation $y(n)=\frac{3}{4}y(n-1)-\frac{1}{8}y(n-2)+x(n)+\frac{1}{3}x(n-1)$ **in cascade form.**

****

Parallel form:

H.W: Realize the system given by difference equation $y(n)=-0.1y(n-1)+0.72y(n-2)+0.7x(n)-0.252x(n-2)$ **in parallel form.**

****

Analog filter design:

There are two types of analog filter design are,

- Butterworth Filter
- Chebyshev Filter.

Analog Low pass Butterworth Filter:

N	Denominator of H(s)
1	$S+1$
2	$s^2+\sqrt{2}s+1$
3	$(s+1)(s^2+s+1)$
4	$(s^2+0.76537s+1)(s^2+1.8477s+1)$
5	$(s+1)(s^2+0.61803s+1)(s^2+1.61803s+1)$
6	$(s^2+1.931855s+1)(s^2+\sqrt{2}s+1)(s^2+0.51764s+1)$
7	$(s+1)(s^2+1.80194s+1)(s^2+1.247s+1)(s^2+0.445s+1)$

Design an analog Butterworth filter that has a -2dB pass band attenuation at a frequency of 20 rad/sec and atleast -10 dB stop band attenuation at 30 rad/sec.

Solution:

Given data:

Pass band attenuation α_P= 2 dB;

Stop band attenuation α_S= 10 dB;

Pass band frequency Ω_P= 20 rad/sec.

Stop band frequency Ω_S=30 rad/sec.

The order of the filter

$$N \geq \left[\frac{\log_{10}\sqrt{\left(\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}\right)}}{\log_{10}\left(\frac{\Omega_s}{\Omega_P}\right)}\right]$$

$$N \geq \left[\frac{\log_{10} \sqrt{\left(\frac{10^{0.1*10} - 1}{10^{0.1*2} - 1} \right)}}{\log_{10}\left(\frac{30}{20} \right)} \right]$$

$$\geq \frac{\log \sqrt{\left(\frac{10-1}{10^{0.2} - 1} \right)}}{\log \frac{30}{20}}$$

$$\geq 3.37$$

Rounding off 'N' to the next higher integer, we get

N=4

The normalized transfer function for *N=4*.

$$H_a(s) = \frac{1}{(s^2 + 0.76537s + 1)(s^2 + 1.8477s + 1)}$$

To find cut off frequency

$$\Omega_c = \frac{\Omega_p}{(10^{0.1\alpha p} - 1)^{1/2N}}$$

$$\Omega_c = \frac{20}{\left(10^{0.1*20} - 1\right)^{\frac{1}{2*4}}} = 21.3868$$

The transfer function for Ωc=21.3868,

$$H(s) = H_a(s)\Big|_{S->\frac{s}{21.3868}}$$

$$H(s) = \frac{1}{(s^2 + 0.76537s + 1)(s^2 + 1.8477s + 1)}\Bigg| s- > \frac{s}{21.3868}$$

$$H(s) = \frac{1}{\left(\frac{s}{21.868}\right)^2 + 0.76535\left(\frac{s}{21.3868}\right) + 1} * \frac{1}{\left(\frac{s}{21.868}\right)^2 + 1.8477\left(\frac{s}{21.3868}\right) + 1}$$

$$H(s) = \frac{0.20921 \times 10^6}{(s^2 + 16.3686s + 457.394)(s^2 + 39.5176s + 457.394)}$$

****

H.W: Challenge 1: For the given specification design an analog Butterworth filter

$$0.9 \leq |H(j\Omega)| \leq 1 \quad for\, 0 \leq \Omega \leq 0.2\pi$$

$$|H(j\Omega)| \leq 0.2 \quad for\, 0.4\pi \leq \Omega \leq \pi.$$

Ans:

$$H(s) = \frac{0.323}{(s^2 + 0.577s + 0.0576\pi^2)(s^2 + 1.393s + 0.0576\pi^2)}$$

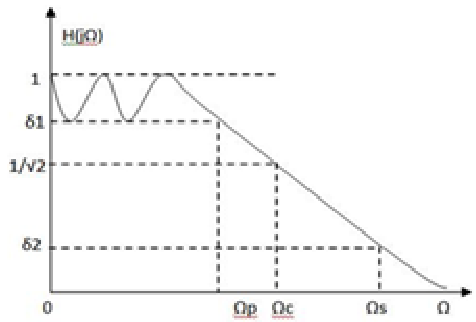
Challenge 2: Determine the order and the poles of low pass Butterworth filter that has 3 dB attenuation at 500 Hz and an attenuation of 40dB at 1000Hz.

Ans: $H(s) = (s+1)(s^2 + 1.80194s + 1)(s^2 + 1.247s + 1)(s^2 + 0.445s + 1)$

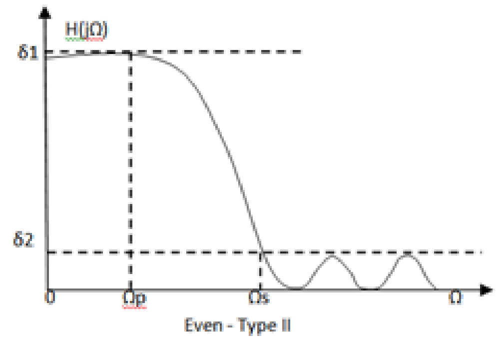
Given the specification $\alpha_p = 1dB; \alpha_s = 30dB; \Omega_p = 200\, rad/\sec; \Omega_s = 600\, rad/sec.$ **determine the order of the filter. Ans:** ***N=4***

Analog Low pass Chebyshev Filter:

There are two types of Chebyshev filters.



Odd - Type I



Even - Type II

Given specifications αp= 3dB,αs=16 dB, f_P=1KHz and f_S=2KHz. Determine the order of the filter using Chebyshev approximation. Find H(s).

Solution:

Given:

Step 1:

Pass band attenuation αp= 3dB,

Stop band attenuation αs=16 dB,

Pass band frequency f_P=1 KHz=2π*1000=2000π rad/sec

Stop band frequency f_S=2 KHz=2π*2*1000=4000π rad/sec

Step 2: Order of the filter

$$N \geq \frac{\cosh^{-1}\sqrt{\left(\frac{10^{0.1\alpha_S}-1}{10^{0.1\alpha_P}-1}\right)}}{\cosh^{-1}\left(\frac{\Omega_S}{\Omega_P}\right)}$$

$$N \geq \frac{\cosh^{-1}\sqrt{\left(\frac{10^{0.1*16}-1}{10^{0.1*3}-1}\right)}}{\cosh^{-1}\left(\frac{4000\pi}{2000\pi}\right)}$$

$$\geq 1.91$$

Rounding the next higher integer value *N=2*.

Step 3: The value of minor axis and major axis can be found as below

$$\varepsilon = \sqrt{\left(10^{0.1*\alpha_P}-1\right)} = \sqrt{\left(10^{0.1*3}-1\right)} = 1$$

$$\mu = \varepsilon^{-1} + \sqrt{1+\varepsilon^{-2}} = 1^{-1} + \sqrt{1+1^{-2}} = 2.414$$

$$a = \Omega_P \frac{\left[\mu^{\frac{1}{N}} - \mu^{-\frac{1}{N}}\right]}{2} = 2000\pi \frac{\left[(2.414)^{\frac{1}{2}} - (2.414)^{-\frac{1}{2}}\right]}{2} = 910\pi$$

$$b = \Omega_S \frac{\left[\mu^{\frac{1}{N}} + \mu^{-\frac{1}{N}}\right]}{2} = 4000\pi \frac{\left[(2.414)^{\frac{1}{2}} + (2.414)^{-\frac{1}{2}}\right]}{2} = 2197\pi$$

Step 4: The poles are given by

$$S_K = a\cos\phi_k + jb\sin\phi_k; \quad k = 1,2$$

$$\phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N}; \quad k = 1,2$$

$For\, k = 1$

$$\phi_1 = \frac{\pi}{2} + \frac{(2-1)\pi}{2*2} = \frac{\pi}{2} + \frac{\pi}{4} = 135^o$$

$Fork = 2$

$$\phi_2 = \frac{\pi}{2} + \frac{(2*2-1)\pi}{2*2} = \frac{\pi}{2} + \frac{3\pi}{4} = 225^o$$

$$s_1 = a\cos\phi_1 + jb\sin\phi_1 = (910\pi * \cos 135) + j(2197 * \sin 135)$$

$$s_1 = -643.46\pi + j1553\pi$$

$$s_2 = a\cos\phi_2 + jb\sin\phi_2 = (910\pi * \cos 225) + j(2197 * \sin 225)$$

$$s_2 = -643.46\pi - j1553\pi$$

Step 5: The denominator of *H(s):*

$$H(s) = (s + 643.46\pi)^2 + (1554)^2$$

Step 6: The numerator of *H(s):*

$substitute, s = 0$

$$H(s) = \frac{(643.46\pi)^2 + (1554\pi)^2}{\sqrt{1+\varepsilon^2}}$$

$$= \frac{(643.46\pi)^2 + (1554\pi)^2}{\sqrt{1+1^2}} = (1414.38)^2\pi^2$$

The transfer function $H(s)=\dfrac{(1414.38)^2\pi^2}{s^2+1287\pi s+(1682)^2\pi^2}$

HW: Challenge 1: Obtain an analog Chebyshev filter transfer function that satisfies the constraints

$$\frac{1}{\sqrt{2}}\le|H(j\Omega)|\le 1;\ for\ 0\le\Omega\le 2$$

$$|H(j\Omega)|\le 0.1\ for\ \Omega\ge 4$$

Ans: $H(s)=\dfrac{2}{(s+0.596)(s^2+0.596s+3.354)}$

2. Design a Chebyshev filter with a maximum pass band attenuation of 2.5dB at Ω_P=20rad/sec and stop band attenuation of 30 dB at Ω_S=50rad/sec.

Ans: ***N=3.*** $H(s)=\dfrac{2265.27}{(s+6.6)(s^2+6.6s+343.2)}$

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3. For the given specifications find the order of the Chebyshev-I filter

$\alpha_P=1.5dB;\alpha_S=10dB;\Omega_P=2rad/Sec;\ \Omega_S=30rad/\sec.$

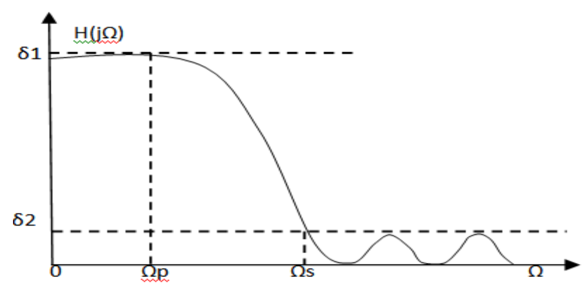
---

4. For the given specifications find the order of the Chebyshev-I filter

$\alpha_P=1dB;\alpha_S=25dB;\Omega_P=1rad/Sec;\ \Omega_S=20rad/\sec.$

<u>Discrete time IIR filter from analog filter:</u>

Magnitude Response of LPF:



Design of IIR filters from analog filters:

The different design techniques available for IIR filter are

1) Approximation of derivates
2) Impulse invariant method
3) Bilinear transformation
4) Matched z-transform techniques.

<u>Approximation of derivates:</u>

For analog to digital domain, we get

$$s=\frac{1-z^{-1}}{T}\ \text{-------------------------- (3)}$$

$$H(z)=H(s)\big|_{s=\frac{1-z^{-1}}{T}}\ \text{--------------------- (4)}$$

Mapping of the s-plane to the z-plane using approximation of derivatives.

Convert the analog BPF with system IIR filter $H_a(s)=\dfrac{1}{(s+0.1)^2+9}$ **into a digital IIR filter by use of the backward difference for the derivative. [Nov/Dec-2015]**

Solution:

Given:

$$H(z) = H_a(s)\Big|_{s=\frac{1-z^{-1}}{T}}$$

$$H(z) = \left.\frac{1}{(s+0.1)^2+9}\right|_{s=\frac{1-z^{-1}}{T}}$$

$$= \frac{1}{\left(\frac{1-z^{-1}}{T}+0.1\right)^2+9}$$

$$= \frac{T^2\left(1+0.2T+9.01T^2\right)}{1-\frac{2(1+0.1T)}{1+0.2T+9.01T^2}z^{-1}+\frac{1}{1+0.2T+9.01T^2}z^{-2}}$$

$T = 0.1\,sec,$

$$= 0.91 \pm j0.27$$

<u>Design of IIR filter using Impulse Invariance Method:</u>

<u>Steps to design a digital filter using Impulse Invariance Method (IIM):</u>

Step 1: For the given specifications, find Ha(s) the Transfer function of an analog filter.

Step 2: Select the sampling rate of the digital filter, T seconds per sample.

Step 3: Express the analog filter transfer function as the sum of single-pole filter.

$$H_a(s) = \sum_{k=1}^{N}\frac{C_k}{S-P_k}$$

Step 4: Compute the z-transform of the digital filter by using formula

$$H(z) = \sum_{k=1}^{N}\frac{C_k}{1-e^{p_kT}z^{-1}}$$

For high sampling rate,

$$\therefore H(z) = \sum_{k=1}^{N}\frac{TC_k}{1-e^{p_kT}z^{-1}}$$

For the analog transfer function $H(s) = \frac{2}{s^2+3s+2}$ **Determine H (z) using impulse invariant transformation if (a) T=1 second and (b) T=0.1 second. [Nov/Dec-15]**

Solution:

$$Given\ that,\ H(s) = \frac{2}{s^2+3s+2} = \frac{2}{(s+1)(s+2)}$$

By partial fraction expansion tchnique we can write,

$$H(s) = \frac{2}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

The roots of quadratic,

$S2+3S+2=0$ *are,*

$$s = \frac{-3 \pm \sqrt{3^2 - 4X2}}{2}$$

$$= \frac{-3 \pm 1}{2} = -1, -2$$

$$A = \frac{2}{\cancel{(s+1)}(s+2)} X \cancel{(s+1)}\Bigg|_{s=-1} = \frac{2}{-1+2} = 2$$

$$B = \frac{2}{(s+1)\cancel{(s+2)}} X \cancel{(s+2)}\Bigg|_{s=-2} = \frac{2}{-2+1} = -2$$

$$\therefore H(s) = \frac{2}{s+1} + \frac{-2}{s+2}$$

By impulse invariant transformation we know that,

$$\frac{A_i}{s+p_i} \rightarrow \frac{A_i}{1-e^{-p_iT}z^{-1}}$$

$$H(z) = \frac{2}{1-e^{-p_iT}z^{-1}} + \frac{-2}{1-e^{-p_iT}z^{-1}}\ Where\ p_1 = 1, p_2 = -2$$

$$\boxed{H(z) = \frac{2}{1-e^{-T}z^{-1}} + \frac{-2}{1-e^{-2T}z^{-1}}}$$

$(a)\,\text{When}\, T = 1\text{second}$

$$H(z) = \frac{2}{1-e^{-1}z^{-1}} + \frac{-2}{1-e^{-2}z^{-1}}$$

$$H(z) = \frac{2}{1-0.1353z^{-1}} + \frac{2}{1-0.1353z^{-1}} = \frac{2(1-0.1353z^{-1}) - 2(1-0.3679z^{-1})}{(1-0.367z^{-1})(1-0.1353z^{-1})}$$

$$= \frac{2-0.27606z^{-1} - 2 + 0.7358z^{-1}}{1-0.1353z^{-1} - 0.3679z^{-1} + 0.0498z^{-2}} = \frac{0.4652z^{-1}}{1-0.5032z^{-1}+0.0498z^{-2}}$$

$$H(z) = \frac{0.4652z^{-1}}{1-0.5032z^{-1}+0.0498z^{-2}} = \frac{0.4652z^{-1}}{z^{-2}(z^2 - 0.5032z + 0.0498)}$$

$$\boxed{H(z) = \frac{0.4652z}{(z^2 - 0.5032z + 0.0498)}}$$

$(b)\, When\, T = 0.1\sec ond$

$$H(z) = \frac{2}{1-e^{-0.1}z^{-1}} + \frac{-2}{1-e^{-0.2}z^{-1}}$$

$$= \frac{2}{1-0.9048z^{-1}} + \frac{-2}{1-0.8187z^{-1}} = \frac{2(1-0\,.8187z^{-1}) - 2(1-0.9048z^{-1})}{(1-0.8187z^{-1})(1-0.9048z^{-1})}$$

$$= \frac{2-1.6374z^{-1} - 2 + 1.8096z^{-1}}{1-0.8187z^{-1} - 0.9048z^{-1} + 0.7408z^{-2}} = \frac{0.1722z^{-1}}{1-1.7235z^{-1} + 0.7408z^{-2}}$$

$$H(z) = \frac{0.1722z^{-1}}{1-1.7235z^{-1}+0.7408z^{-2}} = \frac{0.1722z^{-1}}{z^{-2}(z^2 - 1.7235z + 0.7408)}$$

$$= \frac{0.1722z}{z^2 - 1.7235z + 0.7408}$$

$Since, T < 1, we\, can\, compute\, magnitude\, normalized\, transfer\;\; function, H(z).$

$$H_N(z) = T\;\; X\; H(z) = 0.1\, X\; \frac{0.1722z^{-1}}{1-1.7235z^{-1}+0.7408z^{-2}} = \frac{0.1722z^{-1}}{1-1.7235z^{-1}+0.7408z^{-2}}$$

$$H_N(z) = T\;\; X\; H(z) = 0.1\, X\; \frac{0.1722z}{z^2 - 1.7235z + 0.7408} = \frac{0.172z}{z^2 - 1.7235z + 0.7408}$$

Design a third order Butterworth digital filter using impulse invariant technique. Assume sampling period *T=1 sec.*

Given: For N=3, the transfer function of a normalized Butterworth filter is given by

$$H(s) = \frac{1}{(s+1)(s^2+s+1)}$$

Ans: $H(z) = \frac{1}{1-0.368z^{-1}} + \frac{-1+0.66z^{-1}}{1-0.786z^{-1}+0.368z^{-2}}$

Apply impulse invariant method and find *H(z) for* $H(s)=\frac{s+a}{(s+a)^2+b^2}$

Solution:

Given: The transfer function $H(s)=\frac{s+a}{(s+a)^2+b^2}$

Sampling the function produces

$$h(nT)=\begin{cases} e^{-anT}\cos(bnT) & for\ n\geq 0 \\ 0 & otherwise \end{cases}$$

$$H(z)=\sum_{n=0}^{\infty} e^{-anT}\cos(bnT)z^{-n}$$

$$=\sum_{n=0}^{\infty}\left[e^{-anT}z^{-n}\left(\frac{e^{jbnT}+e^{-jbnT}}{2}\right)\right]$$

$$=\frac{1}{2}\sum_{n=0}^{\infty}\left[\left(e^{-(a-jb)T}z^{-1}\right)^n+\left(e^{-(a+jb)T}z^{-1}\right)^n\right]$$

$$=\frac{1}{2}\left[\frac{1}{1-e^{-(a-jb)T}z^{-1}}+\frac{1}{1-e^{-(a+jb)T}z^{-1}}\right]$$

$$H(z)=\frac{1-e^{-aT}\cos(bT)z^{-1}}{1-2e^{-aT}\cos(bT)z^{-1}+e^{-2aT}z^{-2}}$$

Convert analog filter $H_a(s)=\frac{s+0.1}{(s+0.1)^2+9}$ **into digital IIR filter using impulse invariant method. [Nov/Dec-2015]**

Solution:

Given: Analog filter $H_a(s)=\frac{s+0.1}{(s+0.1)^2+9}$

$$\frac{s+a}{(s+a)^2+b^2}=\frac{1-e^{-aT}\cos(bT)z^{-1}}{1-2e^{-aT}\cos(bT)z^{-1}+e^{-2aT}z^{-2}}$$

$$\frac{s+0.1}{(s+0.1)^2+9}=\frac{1-e^{-aT}\cos(bT)z^{-1}}{1-2e^{-aT}\cos(bT)z^{-1}+e^{-2aT}z^{-2}};T=1\,sec$$

$$\frac{s+0.1}{(s+0.1)^2+9}=\frac{1-e^{-0.1}\cos(3)z^{-1}}{1-2e^{-0.1}\cos(3)z^{-1}+e^{-2*0.1}z^{-2}};$$

$$=\frac{1-0.9048*(-0.9899)z^{-1}}{1+1.791z^{-1}+0.818z^{-2}}$$

$$H(z)=\frac{1+0.89566z^{-1}}{1+1.7915z^{-1}+0.818z^{-2}}$$

Convert analog filter $H_a(s)=\dfrac{6}{(s+0.1)^2+36}$ into digital IIR filter whose system function is given above. The digital filter should have ($\omega r = 0.2\pi$). Use impulse invariant mapping T=1sec.

Solution:

Given: Analog filter $H_a(s)=\dfrac{6}{(s+0.1)^2+36}$

$$\frac{b}{(s+a)^2+b^2}=\frac{e^{-aT}\sin bTz^{-1}}{1-2e^{-aT}\cos bTz^{-1}+e^{-2aT}z^{-2}}$$

$$\frac{6}{(s+0.1)^2+36}=\frac{e^{-0.1}\sin(6)z^{-1}}{1-2e^{-0.1}\cos(6)z^{-1}+e^{-2*0.1}z^{-2}}$$

Assume T = 1 sec.

$$\frac{6}{(s+0.1)^2+36}=\frac{-0.2528z^{-1}}{1-2*0.8687z^{-1}+0.818z^{-2}}$$

$$=\frac{-0.2528z^{-1}}{1-1.7374z^{-1}+0.818z^{-2}}$$

H.W: Challenge 1: An analog filter has a transfer function $H_a(s)=\dfrac{10}{s^2+7s+10}$. Design a digital filter equivalent to this using impulse invariant method for T=0.2 sec. [Nov/Dec-15]

$$\text{Ans}: H(z)=\frac{0.2012z^{-1}}{1-1.0378z^{-1}+0.247z^{-2}}$$

2. An analog filter has a transfer function $H(s)=\dfrac{5}{s^3+6s^2+11s+6}$. Design a digital equivalent to this using impulse invariant method for T=1 sec.

3. An analog filter has a transfer function $H(s)=\dfrac{s+3}{s^2+6s+25}$. Design a digital filter equivalent to this using impulse invariant method T=1 sec.

<u>Design of IIR filters using Bilinear Transformation:</u>

Steps to design digital filter using bilinear transform technique:

1. From the given specifications, find prewarping analog frequencies using formula $\Omega=\dfrac{2}{T}\tan\dfrac{\omega}{2}$
2. Using the analog frequencies find *H(s)* of the analog filter.
3. Select the sampling rate of the digital filter, call it T seconds per sample.
4. Substitute $s=\dfrac{2}{T}\left(\dfrac{1-z^{-1}}{1+z^{-1}}\right)$ into the transfer function found in step2.

Apply bilinear transformation of $H(s)=\dfrac{2}{(s+1)(s+2)}$ with *T=1 sec* and find *H(z)*.*[Nov/Dec-13]*

Solution:

Given: The system function $H(s) = \frac{2}{(s+1)(s+2)}$

$$Substitute\ s = \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right] in\ H(s)\ to\ get\ H(z)$$

$$H(z) = H(s)\Big|_{s=\frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]}$$

$$= \frac{2}{(s+1)(s+2)}\Bigg|_{s=\frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]}$$

Given *T=1 sec.*

$$H(z) = \frac{2}{\left\{2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+1\right\}\left\{2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+2\right\}}$$

$$= \frac{2\left(1+z^{-1}\right)^2}{\left(3-z^{-1}\right)(4)}$$

$$= \frac{\left(1+z^{-1}\right)^2}{6-2z^{-1}}$$

$$H(z) = \frac{0.166(1+z^{-1})^2}{(1-0.33z^{-1})}$$

Using the bilinear transformation, design a high pass filter, monotonic in pass band with cut off frequency of 1000Hz and down 10dB at 350 Hz. The sampling frequency is 5000Hz. [May/June-16]

Solution:

Given: Pass band attenuation $\alpha_P = 3dB$; Stop band attenuation $\alpha_S = 10dB$

Pass band frequency $\omega_P = 2\pi * 1000 = 2000\pi\ rad/sec.$

Stop band frequency $\omega_S = 2\pi * 350 = 700\pi\ rad/sec.$

$$T = \frac{1}{f} = \frac{1}{5000} = 2*10^{-4}\sec.$$

Prewarping the digital frequencies, we have

$$\Omega_P = \frac{2}{T}\tan\frac{\omega_P T}{2} = \frac{2}{2*10^{-4}}\tan\frac{\left(2000\pi*2*10^{-4}\right)}{2} = 10^4\tan(0.2\pi) = 7265\ rad/sec.$$

$$\Omega_S = \frac{2}{T}\tan\frac{\omega_S T}{2} = \frac{2}{2*10^{-4}}\tan\frac{\left(700\pi*2*10^{-4}\right)}{2} = 10^4\tan(0.07\pi) = 2235\ rad/sec.$$

The order of the filter

$$N \geq \frac{\log\sqrt{\frac{10^{0.1\alpha_S}-1}{10^{0.1\alpha_P}-1}}}{\log\frac{\Omega_S}{\Omega_P}}$$

$$= \frac{\log\sqrt{\frac{10^{0.1\times 10}-1}{10^{0.1\times 3}-1}}}{\log\frac{7265}{2235}}$$

$$= \frac{\log(3)}{\log(3.25)} = \frac{0.4771}{0.5118} = 0.932$$

$$N = 1$$

The first order Butterworth filter for Ω_C=1 rad/sec is H(s) = 1/S+1

The high pass filter for $\Omega_C=\Omega_P$=7265 rad/sec can be obtained by using the transformation.

$$S \to \frac{\Omega_C}{s}$$

$$S \to \frac{7265}{s}$$

The transfer function of high pass filter

$$H(s) = \left.\frac{1}{s+1}\right|_{s=\frac{7265}{s}}$$

$$= \frac{s}{s+7265}$$

Using bilinear transformation

$$H(z) = H(s)\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$= \left.\frac{s}{s+7265}\right|_{s=\frac{2}{2*10^{-4}}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$= \frac{10000\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}{10000\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+7265}$$

$$= \frac{0.5792(1-z^{-1})}{1-0.1584z^{-1}}$$

H.W: 1. Determine H(z) that results when the bilinear transformation is applied to *Ha(s)=*

$\frac{s^2+4.525}{s^2+0.692s+0.504}$ **[Nov/Dec-15]** Ans: $H(z) = \frac{1.4479+0.1783z^{-1}+1.4479z^{-2}}{1-1.18752z^{-1}+0.5299z^{-2}}$

2. An analog filter has a transfer function $H(s) = \frac{1}{s^2 + 6s + 9}$,design a digital filter using bilinear transformation method.

Additional Examples:

Design a digital Butterworth filter satisfying the constraints

$$0.707 \le \left|H(e^{j\omega})\right| \le 1 \qquad for\ 0 \le \omega \le \frac{\pi}{2}$$

$$\left|H(e^{jw})\right| \le 0.2 \quad for\ \frac{3\pi}{4} \le \omega \le \pi$$

With *T=1 sec using bilinear transformation.* *[April/May-2015][May/June-14]*

Solution:

Given data:

Pass band attenuation $\alpha_P = 0.707$ *; Pass band frequency* $\omega_P = \frac{\pi}{2}$ *;*

Stop band attenuation $\alpha_S = 0.2$ *; Stops band frequency* $\omega_S = \frac{3\pi}{4}$ *;*

Step 1: Specifying the pass band and stop band attenuation in dB.

$$Pass\ band\ attenuation\ \alpha_P = -20\log\delta 1 = -20\log(0.707) = 3.0116\,dB$$

$$Stop\ band\ attenuation\ \alpha_S = -20\log\delta 2 = -20\log(0.2) = 13.9794\,dB$$

Step2. Choose *T* and determine the analog frequencies (i.e) Prewarp band edge frequency

$$\Omega_P = \frac{2}{T}\tan\left(\frac{\omega_P T}{2}\right) = \frac{2}{1}\tan\left(\frac{\frac{\pi}{2}}{2}\right) = 2\tan\left(\frac{\pi}{4}\right) = 2Rad/Sec$$

$$\Omega_S = \frac{2}{T}\tan\left(\frac{\omega_S T}{2}\right) = \frac{2}{1}\tan\left(\frac{\frac{3\pi}{4}}{2}\right) = 2\tan\left(\frac{3\pi}{8}\right) = 4.828Rad/Sec$$

Step3. To find order of the filter

$$N \ge \left[\frac{\log_{10}\sqrt{\left(\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}\right)}}{\log_{10}\left(\frac{\Omega_s}{\Omega_P}\right)}\right]$$

$$N \geq \frac{\log \sqrt{\left(\frac{10^{0.1*3.01}-1}{10^{0.1*13.97}-1}\right)}}{\log\left(\frac{4.828}{2}\right)}$$

$$\geq \frac{log \sqrt{\left(\frac{0.9998}{23.945}\right)}}{log(2.414)}$$

$$\geq \frac{\log(0.20433)}{\log(2.414)}$$

$$\geq \frac{-0.6896}{0.382}$$

$$\geq 1.8017$$

Rounding the next higher value N=2

Step 4: The normalized transfer function

$$H_a(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$$

Step 5: Cut off frequency

$$\Omega_c = \frac{\Omega_p}{(10^{0.1\alpha p} - 1)^{1/2N}}$$

$$\Omega_c = \frac{2}{\left(10^{0.1*3.01} - 1\right)^{\frac{1}{2*2}}} = \frac{2}{(0.9998)^{\frac{1}{4}}} = 2 Rad / Sec$$

Step 6: To find Transfer function of *H(s):*

$$H(s) = H_a(s)\Big|_{S->\frac{s}{2}}$$

$$H(s) = \frac{1}{s^2 + \sqrt{2}s + 1}\Bigg|_{S \to \frac{s}{2}}$$

$$= \frac{1}{\left(\frac{s}{2}\right)^2 + \sqrt{2}\left(\frac{s}{2}\right) + 1}$$

$$H(s) = \frac{4}{s^2 + 2.828s + 4}$$

Step 7. Apply Bilinear Transformation with to obtain the digital filter

$$H(z) = H(s)\Big|_{S=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$= \frac{4}{s^2 + 2.828s + 4}\Bigg|_{S=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$= \frac{4(1-z^{-1})^2}{4(1-z^{-1})^2 + 5.656(1-z^{-2}) + 4(1+z^{-1})^2}$$

$$H(z) = \frac{0.2929(1+z^{-1})^2}{1+0.1716z^{-2}}$$

Design a digital Butterworth filter satisfying the constraints

$$0.707 \le \left|H(e^{j\omega})\right| \le 1 \qquad for\ 0 \le \omega \le \frac{\pi}{2}$$

$$\left|H(e^{jw})\right| \le 0.2 \quad for\ \frac{3\pi}{4} \le \omega \le \pi$$

With *T=1 sec using Impulse invariant method. [Nov/Dec-13]*

Solution:

Given data:

Pass band attenuation $\alpha_P = 0.707$ *; Pass band frequency* $\omega_P = \frac{\pi}{2}$ *;*

Stop band attenuation $\alpha_S = 0.2$ *; Stops band frequency* $\omega_S = \frac{3\pi}{4}$ *;*

Step 1: Specifying the pass band and stop band attenuation in dB.

Pass band attenuation $\alpha_P = -20\log\delta 1 = -20\log(0.707) = 3.0116dB$

Stop band attenuation $\alpha_S = -20\log\delta 2 = -20\log(0.2) = 13.9794dB$

Step2. Choose *T* and determine the analog frequencies (i.e) Prewarp band edge frequency

$$\omega_P = \Omega_P T = \frac{\pi}{2} Rad/Sec$$

$$\omega_S = \Omega_S T = \frac{3\pi}{4} Rad/Sec$$

Step3. To find order of the filter

$$N \ge \left[\frac{\log_{10}\sqrt{\left(\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}\right)}}{\log_{10}\left(\frac{\Omega_s}{\Omega_P}\right)}\right]$$

$$N \geq \frac{\log \sqrt{\left(\frac{10^{0.1*3.01}-1}{10^{0.1*13.97}-1}\right)}}{\log\left(\frac{\frac{3\pi}{4}}{\frac{\pi}{2}}\right)}$$

$$\geq \frac{log\sqrt{\left(\frac{0.9998}{23.945}\right)}}{log(1.5)}$$

$$\geq \frac{\log(0.20433)}{\log(1.5)}$$

$$\geq \frac{-0.6896}{0.17609}$$

$$\geq 3.924$$

Rounding the next higher value N=4

Step 4: The normalized transfer function

$$H_a(s) = \frac{1}{(s^2 + 0.76537s + 1)(s^2 + 1.8477s + 1)}$$

Step 5: Cut off frequency

$$\Omega_c = \frac{\Omega_p}{(10^{0.1\alpha p} - 1)^{1/2N}}$$

$$\Omega_c = \frac{\frac{\pi}{2}}{(10^{0.1*3.01} - 1)^{\frac{1}{2*4}}} = \frac{\frac{\pi}{2}}{(0.9998)^{\frac{1}{8}}} = 1.57 Rad / Sec$$

Step 6: To find Transfer function of *H(s):*

$$H(s) = H_a(s)\Big|_{S \to \frac{s}{1.57}}$$

$$H(s) = \frac{1}{(s^2 + 0.76537s + 1)(s^2 + 1.8477s + 1)}\Bigg|_{S \to \frac{s}{1.57}}$$

$$= \frac{1}{\left(\left(\frac{s}{1.57}\right)^2 + 0.76537\left(\frac{s}{1.57}\right)+1\right)\left(\left(\frac{s}{1.57}\right)^2 + 1.8477\left(\frac{s}{1.57}\right)+1\right)}\Bigg|$$

$$H(s) = \frac{(1.57)^4}{(s^2 + 1.202s + 2.465)(s^2 + 2.902s + 2.465)}$$

Step 7: Using partial fraction expansion, expand H(s) into

$$H(s) = \frac{A}{(s+1.45+j0.6)} + \frac{A^*}{(s+1.45-j0.6)} + \frac{B}{(s+0.6+1.45j)} + \frac{B^*}{(s+0.6-1.45j)}$$

To find A and A :*

$$A = H(s)\big|_{S=-1.45-j0.6}$$

$$= (s+1.45-j0.6)\frac{(1.57)^4}{(s+1.45+j0.6)(s+1.45-j0.6)(s^2+2.902s+2.465)}\Bigg|_{S=-1.45-j0.6}$$

$$= \frac{(1.57)^4}{(s+1.45-j0.6)(s^2+1.202s+2.465)}\Bigg|_{S=-1.45-j0.6}$$

$$= \frac{(1.57)^4}{(-1.45-j0.6+1.45-j0.6)((-1.45-j0.6)^2+1.202(-1.45-j0.6)+2.465)}$$

$$= \frac{(1.57)^4}{-j(1.2)[1.7425+1.74j-1.7429-j0.7212+2.465]}$$

$$= \frac{(1.57)^4}{-j(1.2)[2.465+j1.0188]} = \frac{5.063}{1.0188-j2.465}$$

$$= \frac{5.063(1.0188-j2.465)}{7.114}$$

$$A = 0.7253 + j1.754;\ A^* = 0.7253 - j1.754$$

To find B and B :*

$$A = H(s)\big|_{S=-0.6-j1.45}$$

$$= (s+0.6+j1.45)\frac{(1.57)^4}{(s+1.45+j0.6)(s+1.45-j0.6)(s+0.6+1.45j)(s+0.6-1.45j)}\Bigg|_{S=-0.6-j1.45}$$

$$= \frac{(1.57)^4}{(-0.6 - j1.45 + 1.45 + j0.6)(-0.6 - j1.45 + 1.45 - j0.6)(-0.6 - j1.45 + 0.6 - 1.45j)}\Bigg|_{S=-0.6-j1.45}$$

$$= \frac{(1.57)^4}{(0.85 - j0.85)(0.85 - j0.85)(-2.9j)}$$

$$= \frac{(1.57)^4}{-j[-1.0187 - j2.468]}$$

$$= \frac{2.095}{-2.468 + j1.0187}$$

$$B = -0.7253 - j0.3; \quad B^* = -0.7253 + j0.3$$

$$H(s) = \frac{0.7253 + j1.754}{s - (-1.45 - j0.6)} + \frac{0.7253 - j1.754}{s - (-1.45 + j0.6)} + \frac{-0.7253 - 0.3j}{s - (-0.6 - j1.45)} + \frac{-0.7253 + 0.3j}{s - (-0.6 + j1.45)}$$

we know for $T = 1$ *sec*

$$H(z) = \sum_{k=1}^{N} \frac{C_k}{1 - e^{p_k} z^{-1}}$$

$$\therefore \quad = \frac{0.7253 + j1.754}{1 - e^{-1.45} e^{-j0.6} z^{-1}} + \frac{0.7253 - j1.754}{1 - e^{-1.45} e^{j0.6} z^{-1}} + \frac{-0.7253 - 0.3j}{1 - e^{-0.6} e^{-j1.45}} + \frac{-0.7253 + 0.3j}{1 - e^{-0.6} e^{j1.45}}$$

$$H(z) = \frac{1.454 + 0.1839z^{-1}}{1 - 0.387z^{-1} + 0.055z^{-2}} + \frac{-1.454 + 0.2307z^{-1}}{1 - 0.1322z^{-1} + 0.301z^{-2}}$$

H.W: Challenge 1: Design a digital Butterworth filter satisfying the constraints

$$0.8 \le \left|H(e^{j\omega})\right| \le 1 \qquad for\ 0 \le \omega \le 0.2\pi$$

$$\left|H(e^{jw})\right| \le 0.2 \quad for\ 0.6\pi \le \omega \le \pi$$

With *T=1 sec using Impulse invariant method.*

Ans: $$H(z) = \frac{0.30109z^{-1}}{1 - 1.048z^{-1} + 0.36z^{-2}}$$

Challenge 2: Design a digital Butterworth filter satisfying the constraints

$$0.8 \le \left|H(e^{j\omega})\right| \le 1 \qquad for\ 0 \le \omega \le 0.2\pi$$

$$\left|H(e^{jw})\right| \le 0.2 \quad for\ 0.6\pi \le \omega \le \pi$$

With *T=1 sec using Bilinear Transformation.*

Ans: $$H(s) = \frac{0.084\left(1 + z^{-1}\right)^2}{1 - 1.028z^{-1} + 0.3651z^{-2}}$$

Challenge 3: Determine the system function H(z) of the lowest order Butterworth digital filter with the following specification.

- ***(a) 3db ripple in pass band*** $0 \le \omega \le 0.2\pi$
- ***(b) 25db attenuation in stop band*** $0.45\pi \le \omega \le \pi$

Ans: $H(z)=\dfrac{0.0687\left(1+z^{-1}\right)^3}{\left(1-0.823z^{-1}\right)\left(1-1.6z^{-1}+0.915z^{-2}\right)}$

Challenge 4: Enumerate the various steps involved in the design of low pass digital Butterworth IIR filter. (ii) The specification of the desired low pass filter is

$$0.8\le\left|H(e^{j\omega})\right|\le 1 \qquad for\ 0\le\omega\le 0.2\pi$$

$$\left|H(e^{jw})\right|\le 0.2 \quad for\ 0.32\pi\le\omega\le\pi$$

Design a Butterworth digital filter using impulse invariant transformation.
Ans:

$$H(z)=\frac{-0.6242+0.2747z^{-1}}{1-0.253z^{-1}+0.5963z^{-2}}+\frac{0.6242-0.1168z^{-1}}{1-1.0358z^{-1}+0.2869z^{-2}}$$

Design a chebyshev filter for the following specification using bilinear transformation.

$$0.8\le\left|H\left(e^{j\omega}\right)\right|\le 1 \qquad 0\le\omega\le 0.2\pi$$

$$\left|H\left(e^{j\omega}\right)\right|\le 0.2 \qquad 0.6\pi\le\omega\le\pi.$$

Solution:

Given data:

Pass band attenuation $\alpha_P=0.8$*; Pass band frequency* $\omega_P=0.2\pi$*;*

Stop band attenuation $\alpha_S=0.2$*; Stops band frequency* $\omega_S=0.6\pi$*;*

Step 1: Specifying the pass band and stop band attenuation in dB.

$$Pass\ band\ attenuation\ \alpha_P=-20\log\delta 1=-20\log(0.8)=1.938dB$$

$$Stop\ band\ attenuation\ \alpha_S=-20\log\delta 2=-20\log(0.2)=13.9794dB$$

Step2. Choose *T* and determine the analog frequencies (i.e) Prewarp band edge frequency

$$\Omega_P=\frac{2}{T}\tan\left(\frac{\omega_P}{2}\right)=2\tan\left(\frac{0.2\pi}{2}\right)=0.649dB$$

$$\Omega_S=\frac{2}{T}\tan\left(\frac{\omega_S}{2}\right)=2\tan\left(\frac{0.6\pi}{2}\right)=2.75dB$$

Step3. To find order of the filter

$$N\ge\frac{Cosh^{-1}\sqrt{\dfrac{10^{0.1\alpha_S}-1}{10^{0.1\alpha_P}-1}}}{Cosh^{-1}\left(\dfrac{\Omega_s}{\Omega_p}\right)}$$

$$\geq \frac{Cosh^{-1}\sqrt{\dfrac{10^{0.1*13.97}-1}{10^{0.1*1.938}-1}}}{Cosh^{-1}\left(\dfrac{2.75}{0.649}\right)}$$

$$\geq \frac{Cosh^{-1}\sqrt{\dfrac{23.945}{0.562}}}{Cosh^{-1}\left(\dfrac{2.75}{0.649}\right)}$$

$$\geq \frac{Cosh^{-1}(6.5273)}{Cosh^{-1}(4.2372)}$$

$$\geq \frac{2.5632}{2.1228}$$

$$\geq 1.207$$

Rounding the next higher integer value N=2

Step4. The poles of chebyshev filter can be determined by

$$S_k = a\cos\varphi_k + jb\sin\varphi_k, k = 0,1,\cdots,N$$

Where,

$$\phi_k = \left[\frac{(2k+N-1)\pi}{2N}\right]$$ And calculate a, b, ε, μ

$$\varepsilon = \sqrt{10^{0.1\alpha p}-1},$$

$$= \sqrt{10^{0.1*1.938}-1}$$

$$\varepsilon = 0.75$$

$$\mu = \varepsilon^{-1} + \left[\sqrt{1+\varepsilon^{-2}}\right]$$

$$= (0.75)^{-1} + \left[\sqrt{1+(0.75)^{-2}}\right]$$

$$\mu = 3$$

$$a = \Omega_p\left[\frac{\mu^{1/N} - \mu^{-1/N}}{2}\right]$$

$$= 0.649\left[\frac{(3)^{\frac{1}{2}} - (3)^{-\frac{1}{2}}}{2}\right]$$

$$a = 0.375$$

$$b = \Omega_p \left[\frac{\mu^{1/N} + \mu^{-1/N}}{2} \right]$$

$$= 0.649 \left[\frac{(3)^{\frac{1}{2}} - (3)^{-\frac{1}{2}}}{2} \right]$$

$$b = 0.75$$

$$\phi_k = \left[\frac{(2k + N - 1)\pi}{2N} \right] ; \quad k = 1,2$$

$$\phi_1 = \left[\frac{(2(1) + 2 - 1)\pi}{2*2} \right] = \frac{3\pi}{4} = 135^0$$

$$\phi_2 = \left[\frac{(2(2) + 2 - 1)\pi}{2*2} \right] = \frac{5\pi}{4} = 225^0$$

$$S_K = a\cos\varphi_k + jb\sin\varphi_k, \quad k = 1,2$$

$$for\, k = 1,$$

$$S_1 = 0.375\cos\varphi_1 + j(0.75)\sin\varphi_1$$

$$= 0.375\cos 135^0 + j(0.75)\sin 135^0$$

$$S_1 = -0.265 + j0.53$$

$$for\, k = 2,$$

$$S_1 = 0.375\cos\varphi_2 + j(0.75)\sin\varphi_2$$

$$= 0.375\cos 225^0 + j(0.75)\sin 225^0$$

$$S_1 = -0.265 - j0.53$$

Step.5 Find the denominator polynomial of the transfer function using above poles.

$$H(s) = \{S + 0.265 - j0.53\}\{S + 0.265 - j0.53\}$$

$$= \{(S + 0.265)^2 - (j0.53)^2\}$$

$$= (S + 0.265)^2 + (0.53)^2$$

$$= S^2 + 0.5306s + 0.3516$$

Step 6 : The numerator of the transfer function depends on the value of N.

- If N is Even substitute s=0 in the denominator polynomial and divide the result by $\sqrt{1+\varepsilon^2}$ Find the value. This value is equal to numerator

$$= \frac{0.3516}{\sqrt{1+\varepsilon^2}} = \frac{0.3516}{\sqrt{1+(0.75)^2}}$$

$$H(s) = 0.28$$

Step 7: The Transfer function is

$$H(s) = \frac{NM}{DM}$$

$$H(s) = \frac{0.28}{s^2 + 0.5306s + 0.3516}$$

Step 8: Apply bilinear transformation with to obtain the digital filter

$$H(z) = H(s)\Big|\, s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

$$H(z) = \frac{0.28}{s^2 + 0.5306s + 0.3516}\Bigg|\, s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

$$= \frac{0.28}{s^2 + 0.5306s + 0.3516}\Bigg|\, s = 2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

$$= \frac{0.28}{\left(2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right)^2 + 0.5306\left(2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right) + 0.3516}$$

$$H(z) = \frac{0.28\left(1+z^{-1}\right)^2}{1 - 1.348z^{-1} + 0.608z^{-2}}$$

==

Design a chebyshev filter for the following specification using impulse invariance method.

$$0.8 \le \left|H\left(e^{j\omega}\right)\right| \le 1 \qquad 0 \le \omega \le 0.2\pi$$

$$\left|H\left(e^{j\omega}\right)\right| \le 0.2 \qquad 0.6\pi \le \omega \le \pi.\ [May/June - 2016]$$

Solution:

Given data:

Pass band attenuation $\alpha_P = 0.8$*; Pass band frequency* $\omega_P = 0.2\pi$ *;*

Stop band attenuation $\alpha_S = 0.2$*; Stops band frequency* $\omega_S = 0.6\pi$ *;*

Step 1: Specifying the pass band and stop band attenuation in dB.

Pass band attenuation $\alpha_P = -20\log\delta 1 = -20\log(0.8) = 1.938dB$

Stop band attenuation $\alpha_S = -20\log\delta 2 = -20\log(0.2) = 13.9794dB$

Step2. Choose *T* and determine the analog frequencies (i.e) Prewarp band edge frequency

$$\Omega_P = \frac{\omega_P}{T} = 0.2\pi\ Rad/Sec$$

$$\Omega_S = \frac{\omega_S}{T} = 0.6\pi\ Rad/Sec$$

Step3. To find order of the filter

$$N \geq \frac{Cosh^{-1}\sqrt{\frac{10^{0.1\alpha_S}-1}{10^{0.1\alpha_P}-1}}}{Cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right)}$$

$$\geq \frac{Cosh^{-1}\sqrt{\frac{10^{0.1*13.97}-1}{10^{0.1*1.938}-1}}}{Cosh^{-1}\left(\frac{0.6\pi}{0.2\pi}\right)}$$

$$\geq \frac{Cosh^{-1}\sqrt{\frac{23.945}{0.562}}}{Cosh^{-1}(3)}$$

$$\geq \frac{Cosh^{-1}(6.5273)}{Cosh^{-1}(3)}$$

$$\geq \frac{2.5632}{1.7627}$$

$$\geq 1.454$$

Rounding the next higher integer value N=2

Step4. The poles of chebyshev filter can be determined by

$$S_k = a\cos\varphi_k + jb\sin\varphi_k, k = 0,1,\cdots,N$$

Where,

$$\phi_k = \left[\frac{(2k+N-1)\pi}{2N}\right]$$ And calculate a, b, ε, μ

$$\varepsilon = \sqrt{10^{0.1\alpha p}-1},$$

$$= \sqrt{10^{0.1*1.938}-1}$$

$$\varepsilon = 0.75$$

$$\mu = \varepsilon^{-1} + \left[\sqrt{1+\varepsilon^{-2}}\right]$$

$$= (0.75)^{-1} + \left[\sqrt{1+(0.75)^{-2}}\right]$$

$$\mu = 3$$

$$a = \Omega_p\left[\frac{\mu^{1/N}-\mu^{-1/N}}{2}\right]$$

$$= 0.2\pi\left[\frac{(3)^{\frac{1}{2}}-(3)^{-\frac{1}{2}}}{2}\right]$$

$$a = 0.362$$

$$b = \Omega_p \left[\frac{\mu^{1/N} + \mu^{-1/N}}{2} \right]$$

$$= 0.2\pi \left[\frac{(3)^{\frac{1}{2}} - (3)^{-\frac{1}{2}}}{2} \right]$$

$$b = 0.7255$$

$$\phi_k = \left[\frac{(2k + N - 1)\pi}{2N} \right] \; ; \quad k = 1,2$$

$$\phi_1 = \left[\frac{(2(1) + 2 - 1)\pi}{2*2} \right] = \frac{3\pi}{4} = 135^0$$

$$\phi_2 = \left[\frac{(2(2) + 2 - 1)\pi}{2*2} \right] = \frac{5\pi}{4} = 225^0$$

$$S_K = a \cos\varphi_k + jb \sin\varphi_k, \quad k = 1,2$$

$$for\, k = 1,$$

$$S_1 = 0.362 \cos\varphi_1 + j(0.7255) \sin\varphi_1$$

$$= 0.362 cos 135^0 + j(0.7255) \sin 135^0$$

$$S_1 = -0.256 + j0.513$$

$$for\, k = 2,$$

$$S_1 = 0.362 \cos\varphi_2 + j(0.7255) \sin\varphi_2$$

$$= 0.362 cos 225^0 + j(0.7255) \sin 225^0$$

$$S_1 = -0.256 - j0.513$$

Step.5 Find the denominator polynomial of the transfer function using above poles.

$$H(s) = \{S + 0.256 - j0.513\}\{S + 0.256 - j0.513\}$$

$$= \{(S + 0.256)^2 - (j0.513)^2\}$$

$$= (S + 0.256)^2 + (0.513)^2$$

$$= S^2 + 0.513s + 0.33$$

Step 6 : The numerator of the transfer function depends on the value of N.

- If N is Even substitute s=0 in the denominator polynomial and divide the result by $\sqrt{1+\varepsilon^2}$ Find the value. This value is equal to numerator

$$= \frac{0.33}{\sqrt{1+\varepsilon^2}} = \frac{0.33}{\sqrt{1+(0.75)^2}}$$

$$H(s) = 0.264$$

Step 7: The Transfer function is

$$H(s) = \frac{NM}{DM}$$

$$H(s)=\frac{0.264}{s^2+0.513s+0.33}$$

Step 8: Using partial fraction expansion, expand H(s) into

$$H(s)=\sum_{k=1}^{2}\frac{A_k}{s-p_k}=\frac{A_1}{s-p_1}+\frac{A_2}{s-p_2}$$

$$\frac{0.264}{s^2+0.513s+0.33}=\frac{A_1}{s-\left(-0.256+j0.514\right)}+\frac{A_2}{s-\left(-0.256-j0.514\right)}$$

$$=\frac{0.257j}{s-\left(-0.256+j0.514\right)}-\frac{0.257j}{s-\left(-0.256-j0.514\right)}$$

Step 9: Now transform analog poles $\{P_k\}$ into digital poles $\{e^{pkT}\}$ to obtain the digital filter

$$H(z)=\sum_{k=1}^{N}\frac{A_k}{1-e^{p_kT}z^{-1}}$$

$$=\sum_{k=1}^{2}\frac{A_k}{1-e^{p_kT}z^{-1}}$$

$$=\frac{0.257j}{s-e^{-0.256T}e^{j0.513T}z^{-1}}-\frac{0.257j}{s-e^{-0.256T}e^{-j0.513T}z^{-1}}$$

$$H(z)=\frac{0.1954z^{-1}}{1-1.3483z^{-1}+0.5987z^{-2}}$$

===

H.W: Challenge 1: Design a chebyshev filter to meet the constraints by using bilinear transformation and assume sampling period T=1 sec.

$$\frac{1}{\sqrt{2}}\le\left|H\left(e^{j\omega}\right)\right|\le 1 \qquad 0\le\omega\le 0.2\pi$$

$$\left|H\left(e^{j\omega}\right)\right|\le 0.1 \qquad 0.5\pi\le\omega\le\pi.$$

Solution:

Ans: $H(z)=\dfrac{0.0413\left(1+z^{-1}\right)^2}{1-1.44z^{-1}+0.675z^{-2}}$

===

H.W: Convert the following analog filter with transfer function $H_a(s)=\dfrac{S+0.2}{(s+0.2)^2+16}$ ***using bilinear transformation. Ans:*** $H(z)=\dfrac{0.105+0.0192z^{-1}-0.0864z^{-2}}{1+1.155z^{-1}+0.9232z^{-2}}$

Find the order and poles of a low pass Butterworth filter that has 3dB bandwidth of 500Hz and attenuation of 40dB at 1kHz.

===

<u>Filter design using frequency translation (HPF, BPF, BRF):</u>

A digital filter can be converted into a digital high pass, band stop or another digital filter. These transformations are given below.

Low pass to Low pass	Low pass to high pass
$z^{-1} \to \frac{z^{-1}-\alpha}{1-\alpha z^{-1}}$ $where\ \alpha = \frac{\sin\left[(\omega_p - \omega_p^{'})/2\right]}{\sin\left[(\omega_p + \omega_p^{'})/2\right]}$ $\omega_p = passband\ \ frequency\ of\ \ lowpass\ \ filter$ $\omega_p^{'} = passband\ \ frequency\ of\ \ new\ lowpass\ \ filter$	$z^{-1} \to -\left[\frac{z^{-1}+\alpha}{1+\alpha z^{-1}}\right]$ $where\ \alpha = -\frac{\cos\left[(\omega_p^{'} + \omega_p)/2\right]}{\cos\left[(\omega_p^{'} - \omega_p)/2\right]}$ $\omega_p = passband\ \ frequency\ of\ \ lowpass\ \ filter$ $\omega_p^{'} = passband\ \ frequency\ of\ \ highpass\ \ filter$
Low pass to Band pass	**Low pass to Band Stop**
$z^{-1} \to \frac{-\left(z^{-2} - \frac{2\alpha k}{1+k} z^{-1} + \frac{k-1}{k+1}\right)}{\frac{k-1}{k+1} z^{-2} - \frac{2\alpha k}{1+k} z^{-1} + 1}$ $where\ \alpha = \frac{\cos\left[(\omega_u + \omega_l)/2\right]}{\cos\left[(\omega_u - \omega_l)/2\right]}$ $k = \cot\left[\frac{\omega_u - \omega_l}{2}\right]\tan\frac{\omega_p}{2}$ $\omega_u = upper\ cutoff\ \ frequency$ $\omega_l = lower\ cutoff\ \ frequency$	$z^{-1} \to \frac{z^{-2} - \frac{2\alpha}{1+k} z^{-1} + \frac{1-k}{1+k}}{\frac{1-k}{1+k} z^{-2} - \frac{2\alpha}{1+k} z^{-1} + 1}$ $where\ \alpha = \frac{\cos\left[(\omega_u + \omega_l)/2\right]}{\cos\left[(\omega_u - \omega_l)/2\right]}$ $k = \tan\left[\frac{\omega_u - \omega_l}{2}\right]\tan\frac{\omega_p}{2}$ $\omega_u = upper\ cutoff\ \ frequency$ $\omega_l = lower\ cutoff\ \ frequency$

==

Analog Domain:

The frequency transformation can be used to design on LPF with different pass band frequency HPF,BPF and BSF from a normalized Low pass filter Ω_C=1 rad/sec.

Low pass to Low pass	Low pass to high pass
$S \to \frac{S}{\Omega_C}$	$S \to \frac{\Omega_C}{S}$
Low pass to band pass	**Low pass to band stop**
$s \to \frac{s^2 + \Omega_l \Omega_u}{s(\Omega_u - \Omega_l)}$ $\Omega_r = \min\{\lvert A\rvert, \lvert B\rvert\}$ $A = \frac{-\Omega_1^{\ 2} + \Omega_l \Omega_u}{\Omega_1(\Omega_u - \Omega_l)}$ $B = \frac{\Omega_2^{\ 2} - \Omega_l \Omega_u}{\Omega_2(\Omega_u - \Omega_l)}$	$s \to \frac{s(\Omega_u - \Omega_l)}{s^2 + \Omega_l \Omega_u}$ $\Omega_r = \min\{\lvert A\rvert, \lvert B\rvert\}$ $A = \frac{\Omega_1(\Omega_u - \Omega_l)}{-\Omega_1^{\ 2} + \Omega_l \Omega_u}$ $B = \frac{\Omega_2(\Omega_u - \Omega_l)}{-\Omega_2^{\ 2} + \Omega_l \Omega_u}$

==

H:W: 1. Design a digital chebyshev filter

$$\frac{1}{\sqrt{2}} \le H\left(e^{j\omega}\right) \le 1 \qquad for\ 0 \le \omega \le 0.2\pi$$

$$0 \le \left|H\left(e^{j\omega}\right) \le 0.1 \qquad for\ 0.5\pi \le \omega \le \pi\right|$$

by using bilinear transformation and assume period T=1 sec.

Ans :

$$N=2; \quad H(s)=\frac{0.212}{s^2+0.4172s+0.3} \quad ; \; H(z)=\frac{0.0413\left(1+z^{-1}\right)^2}{1-1.44z^{-1}+0.675z^{-2}}$$

2. Enumerate the various steps involved in the design of low pass digital Butterworth IIR filter.

$$0.8\le H\left(e^{j\omega}\right)\le 1 \qquad for\; 0\le\omega\le 0.2\pi$$

$$\left\|H\left(e^{j\omega}\right)\right|\le 0.2 \qquad for\; 0.32\pi\le\omega\le\pi\right|$$

Design Butterworth digital filter using impulse invariant transformation.
Ans:

$$N=4; \quad H(s)=\frac{0.2084}{\left(s^2+0.5171s+0.4565\right)\left(s^2+1.2485s+0.4565\right)} \quad ;$$

$$H(z)=\frac{-0.6242+0.2747z^{-1}}{1-0.253z^{-1}+0.5963z^{-2}}+\frac{0.6242-0.1168z^{-1}}{1-1.03582z^{-1}+0.2869z^{-2}}$$

3. Design a chebyshev low pass filter with the specifications $\alpha s=1dB$ ripple in the pass band $0\le\omega\le 0.2\pi, \alpha s=15dB$ ripple in the stop band $0.3\pi\le\omega\le\pi$, using (a) Bilinear transformation (b) Impulse invariance.

(a) Bilinear transformation:
Ans:

$$N=4; \quad H(s)=\frac{0.04381}{\left(s^2+0.1814s+0.4165\right)\left(s^2+0.4378s+0.1180\right)} \quad ;$$

$$H(z)=\frac{0.001836\left(1+z^{-1}\right)^4}{\left(1-1.499z^{-1}+0.8482z^{-2}\right)\left(1-1.5548z^{-1}+0.6493z^{-2}\right)}$$

(b) Impulse invariance:
Ans:

$$N=4; \quad H(s)=\frac{0.03834}{\left(s^2+0.175s+0.391\right)\left(s^2+0.423s+0.11\right)} \quad ;$$

$$H(z)=\frac{-0.083-0.0245z^{-1}}{1-1.49z^{-1}+0.839z^{-2}}+\frac{0.083+0.0238z^{-1}}{1-1.56z^{-1}+0.655z^{-2}}$$

4. Use the backward difference for the derivative to convert the analog low pass filter with system function.

$$H(s)=\frac{1}{s+2}$$

Ans:
$$H(z)=\frac{1}{3-z^{-1}}$$

5. For the analog transfer function determine H(z) using impulse invariant technique. Assume T=1sec.

$$H(s)=\frac{1}{(s+1)(s+2)}$$ **Ans:** $$H(z)=\frac{0.2326z^{-1}}{1-0.5032z^{-1}+0.0498z^{-2}}$$ **[May/Jue-2016]**

6. Determine H(z) using the impulse invariant technique for the analog transfer function.

$$H(s)=\frac{1}{(s+0.5)\left(s^2+0.5s+2\right)}$$

Ans:
$$H(z)=\frac{0.5}{1-0.6065z^{-1}}-0.5\left(\frac{1-0.1385z^{-1}}{1+0.277z^{-1}+0.606z^{-2}}\right)+0.0898\left(\frac{0.7663z^{-1}}{1-0.277z^{-1}+0.606z^{-2}}\right)$$

7. Using bilinear transformation obtain H (z) if $H(s)=\frac{1}{(s+1)^2}$ **and T=0.1s. Ans:** $H(z)=\frac{0.0476(1+z^{-1})^2}{(1-0.9048z^{-1})^2}$

8. **Convert the analog filter with system function** $H(s)=\dfrac{s+0.1}{(s+0.1)^2+9}$ **into a digital IIR filter using bilinear transformation. The digital filter should have a resonant frequency of** $\omega r=\dfrac{\pi}{4}$**. [Nov/Dec-2015]**

$$H(z)=\frac{1+0.027z^{-1}-0.973z^{-2}}{8.572-11.84z^{-1}+8.177z^{-2}}$$

9. **A digital filter with a 3 dB bandwidth of 0.25π is to be designed from the analog filter whose system response is** $H(s)=\dfrac{\Omega c}{s+\Omega c}$ **. Use bilinear transformation and obtain H(z). [Nov/Dec-15]**

$$H(z)=\frac{1+z^{-1}}{3.414-1.414z^{-1}}$$

****

Prove that $\Omega c=\dfrac{\Omega_p}{(10^{0.1\alpha_p}-1)^{1/2N}}=\dfrac{\Omega_s}{(10^{0.1\alpha_s}-1)^{1/2N}}$

Solution:

$$|H(j\Omega)|^2=\frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

$$we\,know\,|H(j\Omega)|^2=\frac{1}{1+\varepsilon^2\left(\frac{\Omega}{\Omega_p}\right)^{2N}}$$

$$by\,comparing$$

$$1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}=1+\varepsilon^2\left(\frac{\Omega}{\Omega_p}\right)^{2N}$$

$$\left(\frac{\Omega}{\Omega_c}\right)^{2N}=\varepsilon^2\left(\frac{\Omega}{\Omega_p}\right)^{2N}$$

$$where\left(\frac{\Omega_p}{\Omega_c}\right)^{2N}=10^{0.1\alpha_p}-1$$

$$there\ fore\ \Omega_c=\frac{\Omega_p}{(10^{0.1\alpha_p}-1)^{1/2N}}=\frac{\Omega_p}{\varepsilon^{1/N}}$$

$$and\left(\frac{\Omega_s}{\Omega_p}\right)^{2N}=\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}$$

$$\Omega_s=\Omega_p\left[\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}\right]^{1/2N}$$

$$=\Omega_c(10^{0.1\alpha_p}-1)^{1/2N}\left[\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}\right]^{1/2N}$$

$$\Omega_c=\frac{\Omega_s}{(10^{0.1\alpha_s}-1)^{1/2N}}$$

$$thus\ \Omega c=\frac{\Omega_p}{(10^{0.1\alpha_p}-1)^{1/2N}}=\frac{\Omega_s}{(10^{0.1\alpha_s}-1)^{1/2N}}$$

UNIT – III FINITE IMPULSE FILTERS

Design of FIR filters - symmetric and Anti-symmetric FIR filters - design of linear phase FIR filters using Fourier series method - FIR filter design using windows (Rectangular, Hamming and Hanning window), Frequency sampling method. FIR filter structures - linear phase structure, direct form realizations

Introduction:

- A digital filter is just a filter that operates on the digital signals.

Types:

- FIR filter design
- IIR filter design

FIR filter:

The digital filter which designed using finite number of response co-efficient is called as finite impulse response filters.

$$h_o(n), h_1(n)....................h_{(N-1)}(n)$$

Advantages:

1. FIR filters have exact linear phase.
2. FIR filters are always stable.
3. FIR filters can be realized in both recursive and non-recursive structure.
4. FIR filters with any arbitrary magnitude response can be tackled using FIR sequence.

Disadvantages:

1. For the same filter specification the order of the FIR filter design can be as high as 5 to 10 times that of an IIR filter.
2. Large storage requirement needed.
3. Powerful computational facilities required for the implementation.

Linear Phase (LP) FIR Filters:

Derive the condition for Linear Phase (LP) FIR Filters. [Nov/Dec-2009]

The transfer function of a FIR causal filter is given by

$$H(\mathrm{z}) = \sum_{n=0}^{N-1} h(n) z^{-n}$$

Where *h(n)* is the impulse response of the filter.

The Fourier transform of h(n) is

$$H(\mathrm{e}^{j\omega}) = \sum_{n=0}^{N-1} h(n) e^{-j\omega n},$$

Which is periodic in frequency with period 2π.

$$H(\mathrm{e}^{j\omega}) = \pm |H(\mathrm{e}^{j\omega})| e^{-j\theta(\omega)}$$

Where $H(\mathrm{e}^{j\omega})$ is magnitude response and $\theta(\ \omega)$ is phase response.

We define the phase delay and group delay of a filter as

$$\tau_p = \frac{-\theta(\omega)}{\omega} and; \tau_g = \frac{-d\theta(\omega)}{d\omega} \text{ -->(1)}$$

For FIR filters with linear phase we can define

$$\theta(\omega) = -\alpha\omega; -\pi \le \omega \le \pi \text{ --->(2)}$$

Where α is a constant phase delay in samples.

Substitute: equation 2 in 1, we have $\tau_p = \tau_g = \alpha$, which means that α is independent of frequency. We can write,

$$\sum_{n=0}^{N-1} h(n) e^{-j\omega n} = \pm |H(e^{j\omega})| e^{-j\theta(\omega)}$$

Which gives us,

$$\sum_{n=0}^{N-1} h(n)\cos\omega n = \pm |H(e^{j\omega})| \cos\theta(\omega) \text{ ------------------------------>(3)}$$

and $$-\sum_{n=0}^{N-1} h(n)\sin\omega n = \pm |H(e^{j\omega})| \sin\theta(\omega) \text{ ---------------------------------->(4)}$$

By taking ratio of equation (3) to equation (4), we obtain

$$\frac{\sum_{n=0}^{N-1} h(n)\sin\omega n}{\sum_{n=0}^{N-1} h(n)\cos\omega n} = \frac{\sin\alpha\omega}{\cos\alpha\omega}; [\theta(\omega) = -\alpha\omega] \text{ ----------------------------->(5)}$$

After simplifying equation (5) we have

$$\sum_{n=0}^{N-1} h(n)\sin(\alpha - \mathrm{n})\omega = 0 \text{ -->(6)}$$

Equation (6) will be zero when

$$h(n) = h(N-1-n) \text{ -->(7)}$$

And $$\alpha = \frac{N-1}{2} \text{ -->(8)}$$

Therefore, FIR filters will have constant phase and group delays when the impulse response is symmetrical about $\alpha = \frac{N-1}{2}$

The impulse response satisfying equation (7) & (8) for odd and even values of N. When N=7 the centre of symmetry of the sequence occurs at third sample and when N=6, the filter delay is $2\frac{1}{2}$ samples.

If only constant group delay is required, and not the phase delay we can write

$$\theta(\omega) = \beta - \alpha\omega$$

Now we have $H(e^{j\omega}) = \pm |H(e^{j\omega})| e^{j(\beta - \alpha\omega)}$

Equation (9) can be expressed as

$$\sum_{n=0}^{N-1} h(n) e^{-j\omega n} = \pm |H(e^{j\omega})| e^{j(\beta-\alpha\omega)} \text{ ------------------------------------->(9)}$$

which gives us

$$\sum_{n=0}^{N-1} h(n)\cos\omega n = \pm |H(e^{j\omega})| \cos(\beta - \alpha\omega) \text{ -------------------------->(10)}$$

and $$-\sum_{n=0}^{N-1} h(n)\sin\omega n = \pm |H(e^{j\omega})| \sin(\beta - \alpha\omega) \text{ ----------------------------->(11)}$$

By taking ratio of equation (11) to (10), we get

$$\frac{-\sum_{n=0}^{N-1} h(n)\sin\omega n}{\sum_{n=0}^{N-1} h(n)\cos\omega n} = \frac{\sin(\beta - \alpha\omega)}{\cos(\beta - \alpha\omega)}$$

From which we obtain

$$\sum_{n=0}^{N-1} h(n)\sin\left[\beta-(\alpha-\text{n})\omega\right]=0 \text{ -->}(12)$$

If $\beta=\frac{\pi}{2}$, Equation (12) becomes,

$$\sum_{n=0}^{N-1} h(n)\cos(\alpha-\text{n})\omega=0 \text{ --->}(13)$$

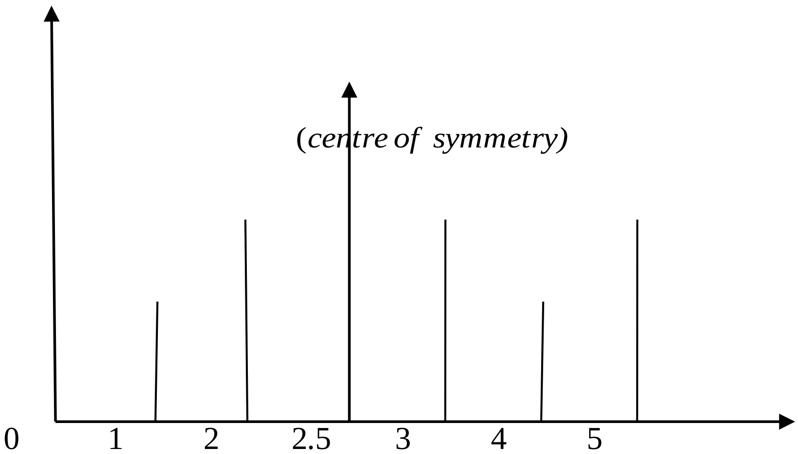
The equation 13 will be satisfied when $h(n)=h(N-1-n)$

And $\alpha=\frac{N-1}{2}$

Therefore, FIR filters have constant group delay, τ_g and not constant phase delay when the impulse response is anti-symmetrical about $\alpha=\frac{N-1}{2}$.

Example:

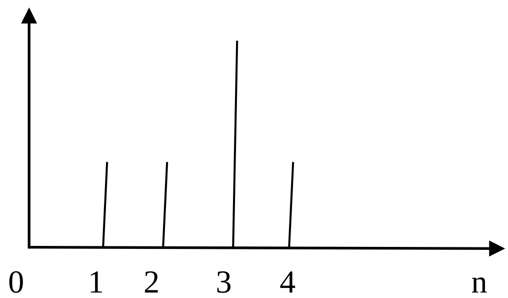
For N=6 $\alpha=\frac{N-1}{2}=\frac{6-1}{2}=2\frac{1}{2}$ *(centre of symmetry)*



For N=7

For N=7 $\alpha=\frac{N-1}{2}=\frac{7-1}{2}=3$*(centre of symmetry)*

centre of symmetry



Linear Phase FIR Filter:

An FIR filter has linear phase if its unit sample response satisfies the condition

$$h(n)=\pm h(M-1-n); \qquad n=0,1,2........N-1$$

<u>Case (i): Symmetric impulse response for "N is ODD":</u>

Determine the frequency response of FIR filter with symmetric impulse response and the order of the filter is "N is Odd".

The frequency response of impulse response can be written as,

$$H\left(e^{j\omega}\right)=\sum_{n=0}^{\frac{M-3}{2}} h(n)e^{-j\omega n}+h\left(\frac{M-1}{2}\right)e^{-\frac{j\omega(M-1)}{2}}+\sum_{n=\frac{N+1}{2}}^{M-1} h(n)e^{-j\omega n} \text{ -------------------------->}(1)$$

$$Let\ n=M-1-n, where\ z=e^{j\omega}\left|H(e^{j\omega})=H(z)\right|z=e^{j\omega}$$

$$H\left(e^{j\omega}\right)= \sum_{n=0}^{\frac{M-3}{2}} h(n)e^{-j\omega n} + h\left(\frac{M-1}{2}\right)e^{-\frac{j\omega(M-1)}{2}} + \sum_{n=0}^{\frac{M-3}{2}} h(M-1-n)\, e^{-j\omega(M-1-n)} \text{ --------->(2)}$$

For a symmetrical impulse response, h(n) = h(M − 1 − n), substituting this relation in above equation (2)

$$H\left(e^{j\omega}\right)= e^{-\frac{j\omega(M-1)}{2}}\left[\sum_{n=0}^{\frac{M-3}{2}} h(n)e^{j\omega\left(\frac{M-1}{2-n}\right)} + \sum_{n=0}^{\frac{M-1}{2}} h(n)\, e^{-\frac{j\omega(M-1)}{2-n}} + h\left(\frac{M-1}{2}\right)\right]$$

$$H\left(e^{j\omega}\right)= e^{-\frac{j\omega(M-1)}{2}}\left[\left[2\sum_{n=0}^{\frac{M-3}{2}} h(n)\cos\omega\left(\frac{M-1}{2}\right)-n\right] + h\left(\frac{M-1}{2}\right)\right]$$

The polar form of $H(e^{j\omega})$ *can be expressed as*

$$H\left(e^{j\omega}\right)=\left|H\left(e^{j\omega}\right)\right|e^{j\angle H(e^{j\omega})}$$

$\therefore$ *Magnitude of* $H(e^{j\omega})$ *is given as* $\quad \left|H(e^{j\omega})\right| = h\left(\frac{M-1}{2}\right) + 2\sum_{n=0}^{\frac{M-3}{2}} h(n)\cos\omega\left(\frac{M-1}{2}-n\right)$

Angle of $H(e^{j\omega})$ *is given as* $\quad \angle H(e^{j\omega}) = \begin{cases} -\omega\left(\frac{M-1}{2}\right), \ for \left|H(e^{j\omega})\right| > 0 \\ -\omega\left(\frac{M-1}{2}\right)+\pi, \ for \left|H(e^{j\omega})\right| < 0 \end{cases}$

==

Case (ii) : Symmetric Impulse Response For –"N is EVEN":

Determine the frequency response of FIR filter with symmetric impulse response and the order of the filter N is Even. [Nov/Dec-2013]

The frequency response of impulse response can be written as,

$$H\left(e^{j\omega}\right)= \sum_{n=0}^{\frac{M-2}{2}} h(n)e^{-j\omega n} + \sum_{n=0}^{\frac{M-2}{2}} h(n)e^{-j\omega n} \text{ -->(1)}$$

Let $n = M-1-n$, *where* $z = e^{j\omega}\left|H(e^{j\omega}) = H(z)\right|z = e^{j\omega}$

$$H\left(e^{j\omega}\right)= \sum_{n=0}^{\frac{M-2}{2}} h(n)e^{-j\omega n} + \sum_{n=0}^{\frac{M-2}{2}} h(M-1-n)\, e^{-j\omega(M-1-n)} \text{ ------------------------------>(2)}$$

For a symmetrical impulse response, h(n) = h(M − 1 − n), substituting this relation in above equation (2)

$$H\left(e^{j\omega}\right)= \sum_{n=0}^{\frac{M-3}{2}} h(n)e^{-j\omega n} + h\left(\frac{M-1}{2}\right)e^{-\frac{j\omega(M-1)}{2}} + \sum_{n=0}^{\frac{M-1}{2}} h(n)\, e^{-j\omega(M-1-n)} \text{ ---------------->(3)}$$

$$H\left(e^{j\omega}\right)= e^{-\frac{j\omega(M-1)}{2}}\left[\sum_{n=0}^{\frac{M-2}{2}} h(n)e^{j\omega\left(\frac{M-1}{2}-n\right)} + \sum_{n=0}^{\frac{M-2}{2}} h(n)\, e^{-j\omega\left(\frac{M-1}{2}-n\right)}\right]$$

$$H\left(e^{j\omega}\right)= e^{-\frac{j\omega(M-1)}{2}}\left[2\sum_{n=0}^{\frac{M-2}{2}} h(n)\cos\omega\left(\frac{M-1}{2}-n\right)\right]$$

The polar form of $H(e^{j\omega})$ *can be expressed as* $H\left(e^{j\omega}\right)=\left|H\left(e^{j\omega}\right)\right|e^{j\angle H(e^{j\omega})}$

$$\therefore \; Magnitude\left|H(e^{j\omega})\right| = 2\sum_{n=0}^{\frac{M-2}{2}} h(n)\cos\omega\left(\frac{M-1}{2}-n\right)$$

$$Angle\ of\ H(e^{j\omega})\ is\ given\ as\ \angle H(e^{j\omega}) = \begin{cases} -\omega\left(\frac{M-1}{2}\right),\ for\left|H(e^{j\omega})\right| > 0 \\ -\omega\left(\frac{M-1}{2}\right)+\pi,\ for\left|H(e^{j\omega})\right| < 0 \end{cases}$$

==

Case (iii) : Antisymmetric for "N is ODD":

Determine the frequency response of FIR filter with Antisymmetric impulse response and the order of the filter N is Odd.

For this type of sequence

$$h\left(\frac{M-1}{2}\right) = 0$$

$$H(e^{j\omega}) = \sum_{n=0}^{M-1} h(n)e^{-j\omega n}$$

The frequency response of impulse response can be written as,

$$H\left(e^{j\omega}\right) = \sum_{n=0}^{\frac{M-3}{2}} h(n)e^{-j\omega n} + \sum_{n=0}^{\frac{M-3}{2}} h(M-1-n)e^{-j\omega(M-1-n)}$$

$$for\ antisymmetric\ impulse\ response, h(n) = -h(M-1-n)$$

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{M-3}{2}} h(n)e^{-j\omega n} - \sum_{n=0}^{\frac{M-3}{2}} h(n)e^{-j\omega(M-1-n)}$$

$$= e^{-j\omega\left(\frac{M-1}{2}\right)}\left[\sum_{n=0}^{\frac{M-3}{2}} h(n)e^{j\omega\left(\frac{M-1}{2}-n\right)} - \sum_{n=0}^{\frac{M-3}{2}} h(n)e^{-j\omega\left(\frac{M-1}{2}-n\right)}\right]$$

$$= e^{-j\omega\left(\frac{M-1}{2}\right)} j\left[2\sum_{n=0}^{\frac{M-3}{2}} h(n)\sin\omega\left(\frac{M-1}{2}-n\right)\right]$$

$$= e^{-j\omega\left(\frac{M-1}{2}\right)} e^{j\frac{\pi}{2}}\left[2\sum_{n=0}^{\frac{M-3}{2}} h(n)\sin\omega\left(\frac{M-1}{2}-n\right)\right]$$

$$H\left(e^{j\omega}\right) = e^{j\left[\frac{\pi}{2}-\omega\left(\frac{M-1}{2}\right)\right]}\left[2\sum_{n=0}^{\frac{M-3}{2}} h(n)\sin\omega\left(\frac{M-1}{2}-n\right)\right]$$

$$Magnitude\ of\ H(e^{j\omega})\ is\ given\ as, \qquad \left|H(e^{j\omega})\right| = 2\sum_{n=0}^{\frac{M-3}{2}} h(n)\sin\omega\left(\frac{M-1}{2}-n\right)$$

$$Angle\ of\ H(e^{j\omega})\ is\ given\ as, \qquad \angle H(e^{j\omega}) = \begin{cases} \frac{\pi}{2}-\omega\left(\frac{M-1}{2}\right)\ for\left|H(e^{j\omega})\right| > 0 \\ \frac{3\pi}{2}-\omega\left(\frac{M-1}{2}\right)\ for\left|H(e^{j\omega})\right| < 0 \end{cases}$$

==

Case (iv) : Antisymmetric For –"N is EVEN":

Determine the frequency response of FIR filter with Antisymmetric impulse response and the order of the filter N is Even. [Nov/Dec-2013]

The frequency response of impulse response can be written as,

$$H\left(e^{j\omega}\right)=\sum_{n=0}^{\frac{M-2}{2}} h(n)e^{-j\omega n}+\sum_{n=0}^{\frac{M-2}{2}} h(M-1-n)e^{-j\omega(M-1-n)}$$

$$\textit{for antisymmetric impulse response}, h(n)=-h(M-1-n)$$

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{M-2}{2}} h(n)e^{-j\omega n} - \sum_{n=0}^{\frac{M-2}{2}} h(n)e^{-j\omega(M-1-n)}$$

$$=e^{-j\omega\left(\frac{M-1}{2}\right)} j\left[2\sum_{n=0}^{\frac{M-2}{2}} h(n)\sin\omega\left(\frac{M-1}{2}-n\right)\right]$$

$$H\left(e^{j\omega}\right)=e^{-j\omega\left(\frac{M-1}{2}\right)} e^{j\frac{\pi}{2}}\left[2\sum_{n=0}^{\frac{M-2}{2}} h(n)\sin\omega\left(\frac{M-1}{2}-n\right)\right]$$

$$H\left(e^{j\omega}\right)=e^{j\left[\frac{\pi}{2}-\omega\left(\frac{M-1}{2}\right)\right]}\left[2\sum_{n=0}^{\frac{M-2}{2}} h(n)\sin\omega\left(\frac{M-1}{2}-n\right)\right]$$

$$=e^{-j\omega\left(\frac{M-1}{2}\right)} j\left[2\sum_{n=0}^{\frac{M-2}{2}} h(n)\sin\omega\left(\frac{M-1}{2}-n\right)\right]$$

$$=e^{j\left[\frac{\pi}{2}-\omega\left(\frac{M-1}{2}\right)\right]}\left[2\sum_{n=0}^{\frac{M-2}{2}} h(n)\sin\omega\left(\frac{M-1}{2}-n\right)\right]$$

$$\textit{Magnitude of } H(e^{j\omega}) \textit{ is given as,} \qquad \left|H(e^{j\omega})\right| = 2\sum_{n=0}^{\frac{M-2}{2}} h(n)\sin\omega\left(\frac{M-1}{2}-n\right)$$

$$\textit{Angle of } H(e^{j\omega}) \textit{ is given as,} \qquad \angle H(e^{j\omega}) = \begin{cases} \frac{\pi}{2}-\omega\left(\frac{M-1}{2}\right) & \textit{for } \left|H(e^{j\omega})\right|>0 \\ \frac{3\pi}{2}-\omega\left(\frac{M-1}{2}\right) & \textit{for } \left|H(e^{j\omega})\right|<0 \end{cases}$$

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Structures of FIR Filters:

Explain with neat sketches the Structure of FIR filters. [Nov/Dec-2012]

The realization of FIR filter is given by

- Transversal structure.
- Linear phase realization
- Polyphase realization.

Transversal structure:

It contains two forms of realization such as,

- Direct form realization
- Cascade form realization.

Direct form realization:

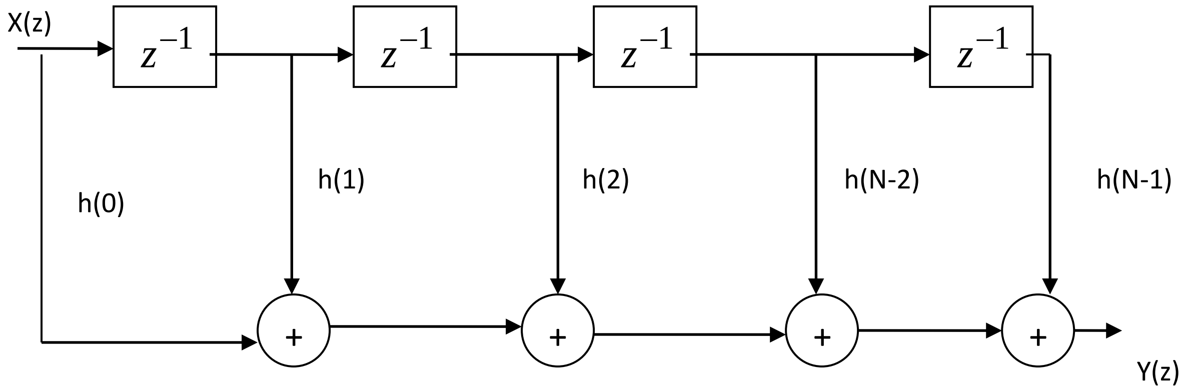
The system function of an FIR filter can be written as

$$H(z)=\sum_{n=0}^{N-1}h(n)\,z^{-n}$$

$$=h(0)+h(1)\,z^{-1}+h(2)\,z^{-2}+\dots\dots\dots+h(N-1)\,z^{-(N-1)} \qquad \text{eq(1)}$$

$$Y(z)=h(0)X(z)+h(1)z^{-1}X(z)+h(2)z^{-2}X(z)+\dots\dots+h(N-1)\,z^{-(N-1)}X(z) \quad \text{eq(2)}$$

This structure is known as direct form realization. It requires N multipliers, N-1 adders, and N-1 delay elements.



Cascade Realization:

Problem 1: Determine the direct form Realization of the following system function. (Nov/Dec-14)

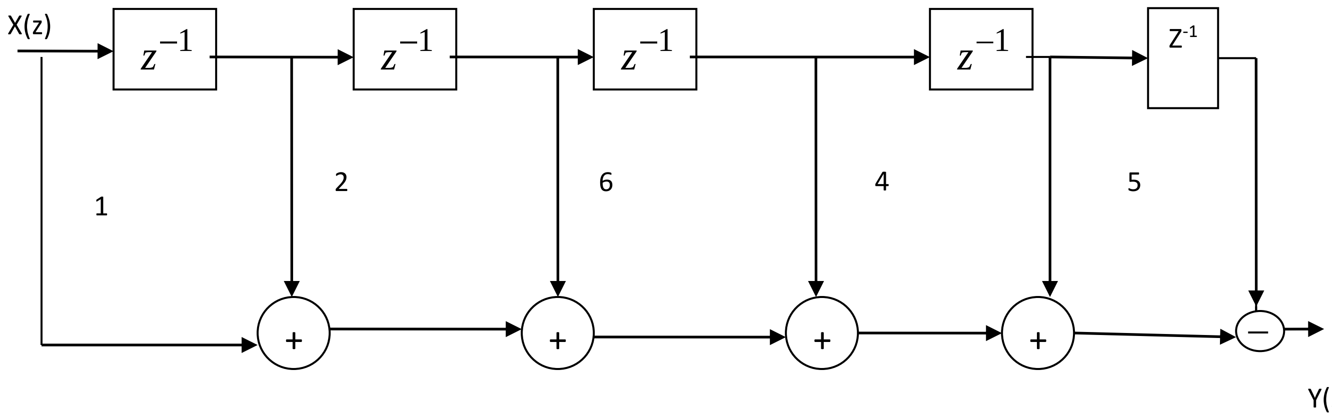
$$H(z)=1+2z^{-1}+6z^{-2}+4z^{-3}+5z^{-4}+8z^{-5}$$

Solution:

Given: The system function is $H(z)=1+2z^{-1}+6z^{-2}+4z^{-3}+5z^{-4}+8z^{-5}$

$$H(z)=\frac{Y(z)}{X(z)}=1+2z^{-1}+6z^{-2}+4z^{-3}+5z^{-4}+8z^{-5}$$

$$Y(z)=X(z)+2z^{-1}X(z)+6z^{-2}X(z)+4z^{-3}X(z)+5z^{-4}X(z)+8z^{-5}X(z)$$



Problem 2: Obtain the cascade realization of system function $H(z)=(1+2z^{-1}-z^{-2})(1+z^{-1}-z^{-2})$ (May/June-12) (Nov/Dec-10)

Solution:

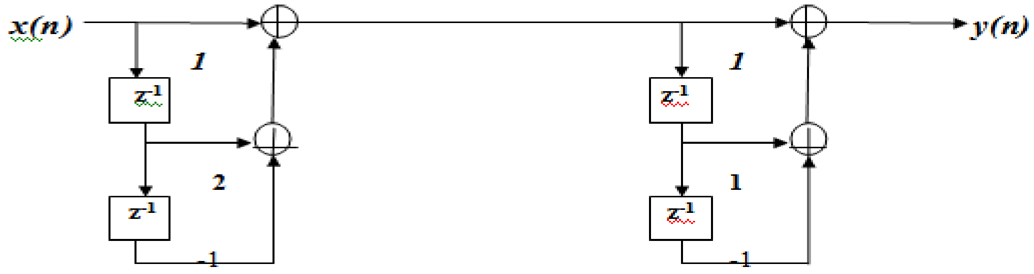
$$H(z) = H_1(z)H_2(z)$$

Where $H_1(z) = 1 + 2z^{-1} - z^{-2}$ and $H_2(z) = 1 + z^{-1} - z^{-2}$

$$H_1(z) = \frac{Y_1(z)}{X_1(z)} \Rightarrow Y_1(z) = X_1(z) + 2z^{-1}X_1(z) - z^{-2}X(z) \qquad eq(1)$$

$$H_2(z) = \frac{Y_2(z)}{X_2(z)} \Rightarrow Y_2(z) = X_2(z) + z^{-1}X_2(z) - z^{-2}X(z) \qquad eq(2)$$

The equation (1) and equation (2) can be realized in direct form and can be cascaded as shown in figure.



H.W 1 : Obtain the direct form realization for the following system function.

1. $H(z) = 1 + 2z^{-1} - 3z^{-2} - 4z^{-3} + 5z^{-4}$

2. $H(z) = \left(1 - \frac{1}{4}z^{-1} + \frac{3}{8}z^{-2}\right)\left(1 - \frac{1}{8}z^{-1} - \frac{1}{2}z^{-2}\right)$

H.W 2: Obtain the cascade form realization for the following system function.

1. $H(z) = 1 + \frac{5}{2}z^{-1} + 2z^{-2} + 2z^{-3}$

2. $H(z) = \left(1 + 2z^{-1}\right)\left(1 + \frac{1}{2}z^{-1} + z^{-2}\right)$

Obtain the linear phase realization of the system function. [Nov/Dec-10]

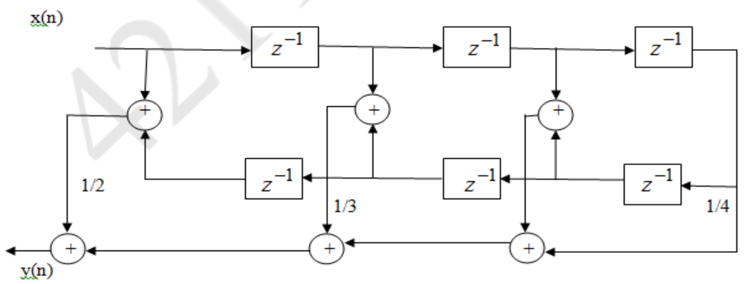
$$H(z) = \frac{1}{2} + \frac{1}{3}z^{-1} + z^{-2} + \frac{1}{4}z^{-3} + z^{-4} + \frac{1}{3}z^{-5} + \frac{1}{2}z^{-6}$$

Solution:

By inspection we find system function H(z) is that of a linear phase FIR filter and,

h(n)=h(N-1-n)

Therefore, we can realize the system function as shown in Figure.



Lattice Structure:

The lattice structure formulas are,

$$\alpha_m(0) = 1$$

$$\alpha_m(m) = k_m$$

$$\alpha_m(k) = \alpha_{m-1}(k) + \alpha_m(m)\alpha_{m-1}(m-1)$$

****

Consider an FIR lattice filter with co-efficients $K_1 = \frac{1}{2}; K_2 = \frac{1}{3}; K_3 = \frac{1}{4}$. Determine the FIR filter direct form structure. [Nov/Dec-2013] [Nov/Dec-2015]

Solution:

Given: The FIR lattice filter with co-efficients are $K_1 = \frac{1}{2}; K_2 = \frac{1}{3}; K_3 = \frac{1}{4}$

$$\alpha_3(0) = 1; \quad \alpha_3(3) = K_3 = \frac{1}{4}$$

$$\alpha_2(2) = K_2 = \frac{1}{3}; \alpha_1(1) = K_1 = \frac{1}{2}$$

We know,

$$\alpha_m(k) = \alpha_{m-1}(k) + \alpha_m(m)\alpha_{m-1}(m-1)$$

For m=2 and K=1

$$\alpha_2(1) = \alpha_1(1) + \alpha_2(2)\alpha_1(1)$$

$$= \frac{1}{2} + \frac{1}{3}.\frac{1}{2} = \frac{2}{3}$$

For m=3 and K=1

$$\alpha_3(1) = \alpha_2(1) + \alpha_3(3)\alpha_2(2)$$

$$= \frac{2}{3} + \frac{1}{4}.\frac{1}{3} = \frac{3}{4}$$

For m=3 and K=2

$$\alpha_3(2) = \alpha_2(2) + \alpha_3(3)\alpha_2(1)$$

$$= \frac{1}{3} + \frac{1}{4}.\frac{2}{3} = \frac{1}{2}$$

$\therefore$ *The lattice filter coeficients are* $\alpha_3(0) = 1; \alpha_3(1) = \frac{3}{4}; \alpha_3(2) = \frac{1}{2}; \alpha_3(3) = \frac{1}{4}$

H.W: Realize the following system in lattice form. [May/June-07]

$$y(n) = 2x(n) + \frac{4}{5}x(n-1) + \frac{3}{2}x(n-2) + \frac{2}{3}x(n-3)$$

Design of FIR Filter:

FIR Filter can be designed using three following techniques.

1. Fourier series method
2. Windowing technique
3. Frequency sampling method.

Filter design using windowing technique:

Explain the designing of FIR filters using windows. [April/May-2011]

The desired frequency response of any digital filter is periodic in frequency and can be expanded in a Fourier series.

$$H_d(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h_d(n)e^{-j\omega n} \quad \text{--->(1)}$$

Where,

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}\, d\omega \quad \text{-->(2)}$$

Gibb's Phenomenon:

One possible way of finding an FIR filter that approximates $H(e^{j\omega})$ would be truncate the infinite Fourier series at $n = \pm\left(\frac{M-1}{2}\right)$. Abrupt truncation of the series will lead to oscillation both pass band and stop band. This phenomenon is known as Gibbs phenomenon.

Types of window:

- Rectangular window.
- Hanning window.
- Hamming window.

Rectangular window:

The rectangular window sequence is given by,

$$w_R(n) = \begin{cases} 1 & for -\left(\frac{M-1}{2}\right) \le n \le \left(\frac{M-1}{2}\right) \\ 0 & for\ otherwise. \end{cases}$$

Hanning window:

The hanning window sequence can be obtained by

$$w_{Hn}(n) = \begin{cases} 0.5 + 0.5\cos\frac{2\pi n}{M-1} & for -\left(\frac{M-1}{2}\right) \le n \le \left(\frac{M-1}{2}\right) \\ 0 & for\ otherwise. \end{cases}$$

Hamming window:

The hamming window can be obtained by

$$w_{Hm}(n) = \begin{cases} 0.54 + 0.46\cos\frac{2\pi n}{M-1} & for -\left(\frac{M-1}{2}\right) \le n \le \left(\frac{M-1}{2}\right) \\ 0 & for\ otherwise. \end{cases}$$

Filter coefficient ($h_d(n)$) for different types of Filters:

Type of Filter	**$h_d(n)$**
LPF	$h_d(n) = \frac{\omega_C}{\pi}\ for\ n = \alpha$ $h_d(n) = \frac{\sin\omega_C(n-\alpha)}{\pi(n-\alpha)}\ for\ n \ne \alpha$
HPF	$h_d(n) = 1 - \frac{\omega_C}{\pi}\ for\ n = \alpha$ $h_d(n) = \frac{1}{\pi(n-\alpha)}\left[\sin(n-\alpha)\pi - \sin(n-\alpha)\omega_C\right] for\ n \ne \alpha$
BPF	$h_d(n) = \frac{\omega_{C2} - \omega_{C1}}{\pi}\ for\ n = \alpha$ $h_d(n) = \frac{1}{\pi(n-\alpha)}\left[\sin\omega_{C2}(n-\alpha) - \sin\omega_{C1}(n-\alpha)\right] for\ n \ne \alpha$
BSF	$h_d(n) = 1 - \left[\frac{\omega_{C2} - \omega_{C1}}{\pi}\right] for\ n = \alpha$ $h_d(n) = \frac{1}{\pi(n-\alpha)}\left[\sin\omega_{C1}(n-\alpha) - \sin\omega_{C2}(n-\alpha) + \sin(n-\alpha)\pi\right] for\ n \ne \alpha$

Design an ideal low pass filter with a frequency response

$$H_d(e^{j\omega}) = \begin{cases} 1 \ for \ -\frac{\pi}{2} \le \omega \le \frac{\pi}{2} \\ 0 \ \ for \ \ \frac{\pi}{2} \le |\omega| \le \pi \end{cases}$$

Find the values of *h(n)* for M=11 using hanning window. Find H(z). Plot the magnitude and frequency response. (May/June-14)(Nov/Dec-14) (April/May 2011)(April/May-08) (Nov/Dec-09) (Nov/Dec-10)

$$0 \ \ for \ \ \frac{\pi}{2} \le |\omega| \le \pi$$

Step 1: To find filter coefficient.

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1.e^{j\omega n}\, d\omega$$

$$= \frac{\sin(n-\alpha)\omega_C}{\pi(n-\alpha)}; \qquad \because \alpha = 0$$

$$h_d(n) = \frac{\sin\frac{\pi}{2}n}{\pi n}; \ -5 \le n \le 5$$

$$For\ n = 0; \qquad h_d(0) = \frac{\sin\frac{\pi}{2}(0)}{\pi(0)}$$

$$= \frac{1}{2}\frac{\sin\frac{\pi n}{2}}{\frac{\pi n}{2}} \qquad\qquad \therefore \lim_{\theta -> 0}\frac{\sin\theta}{\theta} = 1$$

$$h_d(0) = \frac{1}{2}$$

$$For\ n = 1; \qquad h_d(1) = h_d(-1) = \frac{\sin\frac{\pi}{2}(1)}{\pi(1)} = \frac{1}{\pi} = 0.3183$$

$$For\ n = 2; \qquad h_d(2) = h_d(-2) = \frac{\sin\pi}{2\pi} = 0$$

$$For\ n = 3; \qquad h_d(3) = h_d(-3) = \frac{\sin\frac{3\pi}{2}}{3\pi} = -0.106$$

$$For\ n = 4; \qquad h_d(4) = h_d(-4) = \frac{\sin 2\pi}{4\pi} = 0$$

$$For\ n = 5; \qquad h_d(5) = h_d(-5) = \frac{\sin\frac{5\pi}{2}}{5\pi} = 0.06366$$

Step 2: To find hanning window:

$$w_{Hn}(n) = \begin{cases} 0.5 + 0.5\cos\frac{2\pi n}{M-1} & for -\left(\frac{M-1}{2}\right) \le n \le \left(\frac{M-1}{2}\right) \\ 0 \quad for\ otherwise. \end{cases}$$

$$= 0.5 + 0.5\cos\frac{2\pi n}{10} \qquad for -5 \le n \le (5)$$

$$For\ n = 0; \qquad w_{Hn}(0) = 0.5 + 0.5\cos\frac{\pi(0)}{5} = 1$$

$$For\ n = 1; \qquad w_{Hn}(1) = w_{Hn}(-1) = 0.5 + 0.5\cos\frac{\pi(1)}{5} = 0.9045$$

$$For\ n = 2; \qquad w_{Hn}(2) = w_{Hn}(-2) = 0.5 + 0.5\cos\frac{\pi(2)}{5} = 0.6545$$

$$For\ n = 3; \qquad w_{Hn}(3) = w_{Hn}(-3) = 0.5 + 0.5\cos\frac{\pi(3)}{5} = 0.3454$$

$$For\ n = 4; \qquad w_{Hn}(4) = w_{Hn}(-4) = 0.5 + 0.5\cos\frac{\pi(4)}{5} = 0.0954$$

$$For\ n = 5; \qquad w_{Hn}(5) = w_{Hn}(-5) = 0.5 + 0.5\cos\frac{\pi(5)}{5} = 0$$

Step 3: To find filter coefficients using hanning window are

$$h(n) = h_d(n) * w_{Hn}(n) \qquad for -5 \le n \le 5$$

$$h(0) = h_d(0) * w_{Hn}(0) = (0.5) * 1 = 0.5$$

$$h(1) = h_d(1) * w_{Hn}(1) = 0.3183 * 0.9045 = 0.2879$$

$$h(2) = h_d(2) * w_{Hn}(2) = 0 * 0.6545 = 0$$

$$h(3) = h_d(3) * w_{Hn}(3) = -0.106 * 0.3454 = 0.0366$$

$$h(4) = h_d(4) * w_{Hn}(4) = 0.0636 * 0.0954 = 0.00606$$

$$h(5) = h_d(5) * w_{Hn}(5) = 0 * 0.65 = 0$$

Step 4: The transfer function of the filter is given by

$$H(z) = h(0) + \sum_{n=1}^{\frac{M-1}{2}} \left[h(n)\left(z^{n} + z^{-n}\right)\right]$$

$$= 0.5 + \sum_{n=1}^{5} h(n)\left(z^{n} + z^{-n}\right)$$

$$= 0.5 + h(1)\left(z^{1} + z^{-1}\right) + h(2)\left(z^{2} + z^{-2}\right) + h(3)\left(z^{3} + z^{-3}\right) + h(4)\left(z^{4} + z^{-4}\right) + h(5)\left(z^{5} + z^{-5}\right)$$

$$= 0.5 + 0.287z^{1} + 0.287z^{-1} + 0.0366z^{3} + 0.0366z^{-3} + 0.006z^{4} + 0.006z^{-4}$$

Step 5 : The transfer function of the realizable filter is

$$H'(z) = z^{-\frac{(M-1)}{2}} H(z)$$

$$= z^{-5}\left[0.5 + 0.287z^{1} + 0.287z^{-1} + 0.0366z^{3} + 0.0366z^{-3} + 0.006z^{4} + 0.006z^{-4}\right]$$

$$H'(z) = 0.5z^{-5} + 0.287z^{-4} + 0.287z^{-6} + 0.0366z^{-2} + 0.0366z^{-8} + 0.006z^{-1} + 0.006z^{-9}$$

The filter coefficients of causal filter are given by

$$h(0) = h(3) = h(7) = h(10) == 0, h(1) = h(9) = 0.006; h(2) = h(8) = 0.0366; h(4) = h(6) = 0.287; h(5) = 0.5$$

Step 6 : The frequency response is given by

$$\bar{H}(e^{j\omega}) = \sum_{n=0}^{5} a(n)\cos\omega n$$

where,

$$a(0) = h\left(\frac{M-1}{2}\right) = h(5) = 0.5$$

$$a(n) = 2h\left(\frac{M-1}{2} - n\right)$$

$$a(1) = 2h(5-1) = 2h(4) = 2(0.006) = 0.12$$

$$a(2) = 2h(5-2) = 2h(3) = 0$$

$$a(3) = 2h(5-3) = 2h(2) = 2*0.036 = 0.072$$

$$a(4) = 2h(5-4) = 2h(1) = 2*0.006 = 0.12$$

$$a(5) = 2h(5-5) = 2h(0) = 0$$

$$\bar{H}(e^{j\omega}) = \sum_{n=0}^{5} a(n)\cos\omega n$$

$$= a(0) + a(1)\cos\omega + a(2)\cos 2\omega + a(3)\cos 3\omega + a(4)\cos 4\omega + a(5)\cos 5\omega$$

$$= 0.5 + 0.12\cos\omega + 0.072\cos 3\omega + 0.12\cos 4\omega$$

Magnitude in dB is calculated by varying 0 to 10 and tabulated below.

ω(in degree)	0	1	2	3	4	5
$\bar{H}(e^{j\omega})$	0.812	0.8115	0.810	0.8083	0.8054	0.8018
$\lvert H(e^{j\omega})\rvert_{dB}$	-1.8	-1.814	-1.83	-1.85	-1.88	-1.91

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Design an ideal high pass filter with a frequency response

$$H_d(e^{j\omega}) = \begin{cases} 1 & for\ \frac{\pi}{4} \le |\omega| \le \pi \\ 0 & for\ |\omega| \le \frac{\pi}{4} \end{cases}$$

Find the values of *h(n)* for N=11 using hanning window. (May/June-16)(April/May-08)

Solution:

Given :

$$H_d(e^{j\omega}) = \begin{cases} 1 & for\ \frac{\pi}{4} \le |\omega| \le \pi \\ 0 & for\ |\omega| \le \frac{\pi}{4} \end{cases} \qquad Hence\ \omega_C = \frac{\pi}{4}$$

Step 1: To find filter coefficient.

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1.e^{j\omega n}\, d\omega$$

$$h_d(n) = \frac{1}{\pi(n-\alpha)}\left[\sin(n-\alpha)\pi - \sin(n-\alpha)\omega_C\right] for\ n \ne \alpha$$

$$h_d(n) = \frac{1}{\pi n}\left[\sin n\pi - \sin\frac{n\pi}{4}\right]; \quad -5 \le n \le 5$$

Step 2 :

$$For\ n=0 \qquad h_d(0)=\frac{1}{\pi(0)}\left[\sin(0)\pi-\sin\frac{(0)\pi}{4}\right]$$

$$U\sin g\ ,\ L'\,hospital\ rule \qquad h_d(0)=1-\frac{1}{4}=0.75$$

$$For\ n=1; \qquad h_d(1)=h_d(-1)=\frac{1}{\pi(1)}\left[\sin(1)\pi-\sin\frac{(1)\pi}{4}\right]=-0.225$$

$$For\ n=2; \qquad h_d(2)=h_d(-2)=\frac{1}{\pi(2)}\left[\sin(2)\pi-\sin\frac{(2)\pi}{4}\right]=-0.159$$

$$For\ n=3; \qquad h_d(3)=h_d(-3)=\frac{1}{\pi(3)}\left[\sin(3)\pi-\sin\frac{(3)\pi}{4}\right]=-0.075$$

$$For\ n=4; \qquad h_d(4)=h_d(-4)=\frac{1}{\pi(4)}\left[\sin(4)\pi-\sin\frac{(4)\pi}{4}\right]=0$$

$$For\ n=5; \qquad h_d(5)=h_d(-5)=\frac{1}{\pi(5)}\left[\sin(5)\pi-\sin\frac{(5)\pi}{4}\right]=0.045$$

Step 2: Using Hanning window:

$$w_{Hn}(n)=\begin{cases}0.5+0.5\cos\dfrac{2\pi n}{M-1}, & \left(\dfrac{M-1}{2}\right)\le n\le\left(\dfrac{M-1}{2}\right)\\ 0 & otherwise\end{cases}$$

$$w_{Hn}(0)=0.5+0.5=1$$

$$w_{Hn}(1)=w_{Hn}(-1)=0.5+0.5\cos\left(\frac{\pi(1)}{5}\right)=0.9045$$

$$w_{Hn}(2)=w_{Hn}(-2)=0.5+0.5\cos\left(\frac{\pi(2)}{5}\right)=0.655$$

$$w_{Hn}(3)=w_{Hn}(-3)=0.5+0.5\cos\left(\frac{\pi(3)}{5}\right)=0.345$$

$$w_{Hn}(4)=w_{Hn}(-4)=0.5+0.5\cos\left(\frac{\pi(4)}{5}\right)=0.0945$$

$$w_{Hn}(5)=w_{Hn}(-5)=0.5+0.5\cos\left(\frac{\pi(5)}{5}\right)=0$$

Step 3: The filter coefficients using hanning window are,

$$h(n)=h_d(n)w_{Hn}(n) \quad for\ -5\le n\le 5$$

$$h(0)=h_d(0)w_{Hn}(0)=(0.75)(0)=0$$

$$h(1)=h_d(1)w_{Hn}(1)=(-0.225)(0.905)=-0.204$$

$$h(2)=h_d(2)w_{Hn}(2)=(-0.159)(0.655)=-0.104$$

$$h(3)=h_d(3)w_{Hn}(3)=(-0.075)(0.345)=-0.026$$

$$h(4)=h_d(4)w_{Hn}(4)=(0)(0.8145)=0$$

$$h(5)=h_d(5)w_{Hn}(5)=(0.045)(0)=0$$

Step 4: The transfer function of the filter is given by

The transfer function of the realizable filter is

$$H'(z) = z^{-\left(\frac{M-1}{2}\right)}H(z)$$

$$= z^{-5}\left[0.75-0.204z^{-1}-0.204z^{1}-0.104z^{-2}-0.104z^{2}-0.026z^{-3}-0.026z^{3}\right]$$

$= 0.75z^{-5} - 0.204z^{-6} - 0.204z^{-4} - 0.104z^{-7} - 0.104z^{-3} - 0.026z^{-8} - 0.026z^{-2} H$

$$H(z) = h(0) + \sum_{n=1}^{\frac{M-1}{2}} h(n)\left(z^{-n} + z^{n}\right)$$

$$= h(0) + \sum_{n=1}^{5} h(n)\left[z^{-n} + z^{n}\right]$$

$$= 0.75 + h(1)\left[z^{-1} + z^{1}\right] + h(2)\left[z^{-2} + z^{2}\right] + h(3)\left[z^{-3} + z^{3}\right] + h(4)\left[z^{-4} + z^{4}\right] + h(5)\left[z^{-5} + z^{5}\right]$$

$$= 0.75 - 0.204z^{-1} - 0.204z^{1} - 0.104z^{-2} - 0.104z^{2} - 0.026z^{-3} - 0.026z^{3}$$

The causal filter coefficients are

$h(0) = h(1) = h(9) = h(10) = 0;$

$h(2) = h(8) = -0.026$

$h(3) = h(7) = -0.104$

$h(4) = h(6) = -0.204$

$h(5) = 0.75$

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{M-1}{2}} a(n)\cos\omega n$$

$$a(0) = h\left(\frac{M-1}{2}\right) = h(5) = 0.75$$

$$a(n) = 2h\left[\frac{M-1}{2} - n\right] = 2h(5-n)$$

$a(1) = 2h(5-1) = 2h(4) = -0.408$

$a(2) = 2h(5-2) = 2h(3) = -0.208$

$a(3) = 2h(5-3) = 2h(2) = -0.052$

$a(4) = 2h(5-4) = 2h(1) = 0$

$a(5) = 2h(5-5) = 2h(0) = 0$

$$\bar{H}(e^{j\omega}) = 0.75 - 0.408\cos\omega - 0.208\cos 2\omega - 0.052\cos 3\omega$$

ω *(in degrees)*	0	1	2	3	4	5
$\bar{H}(e^{j\omega})$	0.082	0.0822	0.083	0.08433	0.08615	0.08848
$\lvert H(e^{j\omega})\rvert_{dB}$	-21.72	-21.70	-21.61	-21.480	-21.29	-21.11

b) Using Hamming window:

The hamming window sequence is given by

$$w_{Hm}(n) = \begin{cases} 0.54 + 0.46\cos\left(\frac{2\pi n}{M-1}\right); & for -\left(\frac{M-1}{2}\right) \le n \le \left(\frac{M-1}{2}\right) \\ 0 & otherwise \end{cases}$$

$$w_{Hm}(n) = \begin{cases} 0.54 + 0.46\cos\left(\frac{\pi n}{5}\right); & for -5 \le n \le 5 \\ 0 & otherwise \end{cases}$$

$w_{Hm}(0) = 1$

$$w_{Hm}(1) = w_{Hm}(-1) = 0.912$$
$$w_{Hm}(2) = w_{Hm}(-2) = 0.682$$
$$w_{Hm}(3) = w_{Hm}(-3) = 0.398$$
$$w_{Hm}(4) = w_{Hm}(-4) = 0.1678$$
$$w_{Hm}(5) = w_{Hm}(-5) = 0.08$$

The coefficients using hamming window sequence are,

$$h(n) = h_d(n) * w_{Hm}(n); \ -5 \le n \le 5$$
$$h(0) = h_d(0) * w_{Hm}(0) = 1 * 0.75 = 0.75$$
$$h(1) = h_d(1) * w_{Hm}(1) = (-0.225)*(0.912) = -0.2052$$
$$h(2) = h_d(2) * w_{Hm}(2) = (-0.159)*(0.682) = -0.1084$$
$$h(3) = h_d(3) * w_{Hm}(3) = (-0.075)*(0.398) = -0.03$$
$$h(4) = h_d(4) * w_{Hm}(4) = (0)*(0.16787) = 0$$
$$h(5) = h_d(5) * w_{Hm}(5) = (-0.045)*(0.08) = 0.0036$$

The transfer function of the filter is given by

$$H(z) = h(0) + \sum_{n=1}^{\frac{M-1}{2}} h(n)\left[z^{-n} + z^{n}\right]$$
$$= h(0) + \sum_{n=1}^{5} h(n)\left[z^{-n} + z^{n}\right]$$
$$= 0.75 + h(1)\left[z^{-1} + z^{1}\right] + h(2)\left[z^{-2} + z^{2}\right] + h(3)\left[z^{-3} + z^{3}\right] + h(4)\left[z^{-4} + z^{4}\right] + h(5)\left[z^{-5} + z^{5}\right]$$
$$= 0.75 - 0.2052z^{-1} - 0.2052z^{1} - 0.1084z^{-2} - 0.1084z^{2} - 0.03z^{-3} - 0.03z^{3} + 0.0036z^{-5} + 0.0036z^{5}$$

The transfer function of the realizable filter is

$$H'(z) = z^{-\left(\frac{M-1}{2}\right)} H(z)$$
$$= z^{-5}\left[0.75 - 0.2052z^{-1} - 0.2052z^{1} - 0.1084z^{-2} - 0.1084z^{2} - 0.03z^{-3} - 0.03z^{3} + 0.0036z^{-5} + 0.0036z^{5}\right]$$
$$H'(z) = 0.75z^{-5} - 0.2052z^{-6} - 0.2052z^{-4} - 0.1084z^{-7} - 0.1084z^{-3} - 0.03z^{-8} - 0.03z^{-2} + 0.0036z^{-10} + 0.0036$$

The filter coefficients of causal filter are

$h(0) = h(10) = 0.0036; h(1) = h(9) = 0; h(2) = h(8) = -0.03; h(3) = h(7) = -0.1084; h(4) = h(6) = -0.2052;$
$h(5) = 0.75$

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{M-1}{2}} a(n)\cos\omega n$$

$$a(0) = h\left(\frac{M-1}{2}\right) = h(5) = 0.75$$
$$a(n) = 2h\left(\frac{M-1}{2} - n\right) = 2h(5-n)$$
$$a(1) = 2h(5-1) = 2h(4) = -0.4104$$
$$a(2) = 2h(5-2) = 2h(3) = -0.2168$$
$$a(3) = 2h(5-3) = 2h(2) = -0.06$$
$$a(4) = 2h(5-4) = 2h(1) = 0$$
$$a(5) = 2h(5-5) = 2h(0) = 0.0072$$

$\bar{H}(e^{j\omega}) = 0.75 - 0.4104\cos\omega - 0.2168\cos 2\omega - 0.06\cos 3\omega + 0.0072\cos 5\omega$

ω *(in degrees)*	**0**	**30**	**60**	**90**	**120**	**150**	**180**
$\bar{H}(e^{j\omega})$	**0.07**	**0.28**	**0.7168**	**0.9668**	**1**	**1.003**	**1.0108**
$\lvert H(e^{j\omega})\rvert_{dB}$	**-23.1**	**-11**	**-2.89**	**-0.29**	**0**	**0.028**	**0.093**

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H.W: 1. Design a filter with $H_d(e^{j\omega}) = \begin{cases} e^{-j3\omega}, & -\frac{\pi}{4} \le \omega \le \frac{\pi}{4} \\ 0 & \frac{\pi}{4} \le |\omega| \le \pi \end{cases}$ **. Using a Hamming & Hanning window with N=7**

2. Design a filter with $H_d(e^{j\omega}) = \begin{cases} e^{-j5\omega}, & -\frac{\pi}{2} \le \omega \le \frac{\pi}{2} \\ 0 & \frac{\pi}{2} \le |\omega| \le \pi \end{cases}$ **. Using a Hamming & Hanning window with N=11.**

3. Design an FIR filter for the ideal frequency response using hamming window with N=7.

$$H_d(e^{j\omega}) = \begin{cases} e^{-j3\omega}, & |\omega| \le \frac{\pi}{8} \\ 0 & \frac{\pi}{8} \le |\omega| \le \pi \end{cases}$$

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For a FIR linear phase digital filter approximating the ideal frequency response

$$H_d(e^{j\omega}) = \begin{cases} 1, & |\omega| \le \frac{\pi}{6} \\ 0 & \frac{\pi}{6} \le |\omega| \le \pi \end{cases}$$

Determine the coefficients of a 5 tap filter using rectangular window.

Solution:

Given:

$$H_d(e^{j\omega}) = \begin{cases} 1, & |\omega| \le \frac{\pi}{6} \\ 0 & \frac{\pi}{6} \le |\omega| \le \pi \end{cases} \qquad \textit{Hence } \omega_C = \frac{\pi}{6};\ N = 5.$$

Step 1: To find filter coefficient.

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1.e^{j\omega n}\, d\omega$$

$$= \frac{\sin(n-\alpha)\omega_C}{\pi(n-\alpha)}; \qquad \because \alpha = 0$$

$$h_d(n) = \frac{\sin \frac{\pi}{6} n}{\pi n}; \quad -2 \le n \le 2$$

For n=0: $$h_d(0) = \frac{\sin \frac{\pi}{6}(0)}{\pi(0)}$$

$$= \frac{1}{6} \frac{\sin \frac{\pi n}{6}}{\frac{\pi n}{6}}$$

$$h_d(0) = \frac{1}{6} = 0.16 \qquad \therefore \lim_{\theta -> 0} \frac{\sin \theta}{\theta} = 1$$

$$for \; n=1 \qquad h_d(1) = h_d(-1) = \frac{\sin \frac{\pi}{6}(1)}{\pi(1)} = \frac{0.5}{\pi} = 0.159$$

$$For \; n=2; \qquad h_d(2) = h_d(-2) = \frac{\sin \frac{\pi(2)}{6}}{2\pi} = 0.1379$$

Step 2: Using Rectangular window:

$$w_R(n) = \begin{cases} 1 & for -\left(\frac{M-1}{2}\right) \le n \le \left(\frac{M-1}{2}\right) \\ 0 & for\ otherwise. \end{cases}$$

$$= 1 \quad for -2 \le n \le 2$$

$$w_R(0) = w_R(1) = w_R(2) = 1$$

Step 3: To find filter coefficients using rectangular window are

$$h(n) = h_d(n) * w_{Hn}(n) \quad for -2 \le n \le 2$$

$$h(0) = h_d(0) * w_{Hn}(0) = (0.16)*1 = 0.16$$

$$h(1) = h_d(1) * w_{Hn}(1) = 0.59*1 = 0.59$$

$$h(2) = h_d(2) * w_{Hn}(2) = 0.137*1 = 0.137$$

Step 4: The transfer function of the filter is given by

$$H(z) = h(0) + \sum_{n=1}^{\frac{M-1}{2}} \left[h(n)\left(z^n + z^{-n}\right) \right]$$

$$= 0.5 + \sum_{n=1}^{2} h(n)\left(z^n + z^{-n}\right)$$

$$= 0.16 + h(1)\left(z^1 + z^{-1}\right) + h(2)\left(z^2 + z^{-2}\right)$$

$$= 0.16 + 0.59z^1 + 0.59z^{-1} + 0.137z^{-2} + 0.137z^2$$

Step 5 : The transfer function of the realizable filter is

$$H'(z) = z^{-\frac{(M-1)}{2}} H(z)$$

$$= z^{-2}\left[0.16 + 0.59z^1 + 0.59z^{-1} + 0.137z^{-2} + 0.137z^2\right]$$

$$H'(z) = 0.16z^{-2} + 0.59z^{-1} + 0.59z^{-3} + 0.137z^{-4} + 0.137$$

The filter coefficients of causal filter are given by

$$h(0) = h(4) = 0.137, h(1) = h(3) = 0.59; h(2) = 0.16; h(5) = 0$$

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{M-1}{2}} a(n)\cos\omega n$$

$$a(0) = h\left(\frac{M-1}{2}\right) = h(2) = 0.32$$

$$a(n) = 2h\left(\frac{M-1}{2} - n\right) = 2h(2-n)$$

$$a(1) = 2h(2-1) = 2h(1) = 1.18$$

$$a(2) = 2h(2-2) = 2h(0) = 0.274$$

$$H(e^{j\omega}) = 0.32 + 1.18\cos\omega + 0.274\cos 2\omega$$

ω *(in degrees)*	**0**	**30**	**60**	**90**	**120**	**150**	**180**
$\bar{H}(e^{j\omega})$	**1.774**	**1.47**	**0.773**	**0.046**	**-0.407**	**-0.564**	**-0.586**

Design an ideal band pass filter with a frequency response

$$H_d(e^{j\omega}) = \begin{cases} 1, & \frac{\pi}{4} \le |\omega| \le \frac{3\pi}{4} \\ 0 & otherwise \end{cases}$$

Find the values of *h(n)* for N=11 using rectangular window.

Solution:

Given:

$$H_d(e^{j\omega}) = \begin{cases} 1, & \frac{\pi}{4} \le |\omega| \le \frac{3\pi}{4} \\ 0 & otherwise \end{cases}$$

$$\omega_{C1} = \frac{\pi}{4} \text{ and } \omega_{C2} = \frac{3\pi}{4}$$

Step 1: Filter coefficients are,

$$h_d(n) = \frac{1}{\pi(n-\alpha)}\left[\sin\omega_{C2}(n-\alpha) - \sin\omega_{C1}(n-\alpha)\right] for\ n \ne \alpha$$

$$= \frac{1}{\pi n}\left[\sin\frac{3\pi n}{4} - \sin\frac{\pi n}{4}\right]$$

For n = 0;

$$h_d(0) = 0.5$$

$$h_d(1) = h_d(-1) = \frac{\sin\frac{3\pi(1)}{4} - \sin\frac{\pi(1)}{4}}{\pi(1)} = 0$$

$$h_d(2) = h_d(-2) = \frac{\sin\frac{3\pi(2)}{4} - \sin\frac{\pi(2)}{4}}{\pi(2)} = -0.3183$$

$$h_d(3) = h_d(-3) = \frac{\sin\frac{3\pi(3)}{4} - \sin\frac{\pi(3)}{4}}{\pi(3)} = 0$$

$$h_d(4) = h_d(-4) = \frac{\sin\frac{3\pi(4)}{4} - \sin\frac{\pi(4)}{4}}{\pi(4)} = 0$$

$$h_d(5) = h_d(-5) = \frac{\sin\frac{3\pi(5)}{4} - \sin\frac{\pi(5)}{4}}{\pi(5)} = 0$$

Step 2: Using rectangular window

$$w_R(n) = \begin{cases} 1 & for -\left(\frac{M-1}{2}\right) \le n \le \left(\frac{M-1}{2}\right) \\ 0 & for\ otherwise. \end{cases}$$

$$= 1 \quad for -2 \le n \le 2$$

$$w_R(0) = w_R(1) = w_R(2) = w_R(3) = w_R(4) = w_R(5) = 1$$

Step 3: Filter coefficients using rectangular window

$$h(n) = w_R(n) * h_d(n); \quad -5 \le n \le 5$$

$$h(0) = w_R(0) * h_d(0) = 1 * 0.5 = 0.5$$

$$h(1) = w_R(1) * h_d(1) = 1 * 0 = 0$$

$$h(2) = w_R(2) * h_d(2) = 1 * -0.3183 = -0.3183$$

$$h(3) = w_R(3) * h_d(3) = 1 * 0 = 0$$

$$h(4) = w_R(4) * h_d(4) = 1 * 0 = 0$$

$$h(5) = w_R(5) * h_d(5) = 1 * 0 = 0$$

Step 4: The transfer function of the filter is

$$H(z) = h(0) + \sum_{n=1}^{\frac{M-1}{2}} h(n)\left[z^{-n} + z^{n}\right]$$

$$= 0.5 - 0.3183z^{2} - 0.3183z^{-2}$$

Step 5: The transfer function of the realizable filter is

$$H'(z) = z^{-5}\left(0.5 - 0.3183z^{2} - 0.3183z^{-2}\right)$$

$$= 0.5z^{-5} - 0.3183z^{-3} - 0.3183z^{-7}$$

The filter coefficients of the causal filers are

$$h(0) = h(10) = h(9) = h(2) = h(8) = h(4) = h(6) = 0$$

$$h(3) = h(7) = -0.3183$$

$$h(5) = 0.5$$

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{M-1}{2}} a(n)\cos\omega n$$

$$a(0) = h\left(\frac{M-1}{2}\right) = h(5) = 0.5$$

$$a(n) = 2h\left(\frac{M-1}{2} - n\right) = 2h(5-n)$$

$$a(1) = 2h(5-1) = 2h(4) = 0$$

$$a(2) = 2h(5-2) = 2h(3) = -0.6366$$

$$a(3) = 2h(5-3) = 2h(2) = 0$$

$$a(4) = 2h(5-4) = 2h(1) = 0$$

$$a(5) = 2h(5-5) = 2h(0) = 0$$

$$\bar{H}(e^{j\omega}) = 0.5 - 0.6366\cos 2\omega$$

ω *(in degrees)*	**0**	**30**	**60**	**90**	**120**	**150**	**180**
$\bar{H}(e^{j\omega})$	**-0.1366**	**0.1817**	**0.818**	**1.1366**	**0.818**	**0.1817**	**-0.1366**
$\left\lvert H(e^{j\omega})\right\rvert_{dB}$	**-17.3**	**-14.8**	**-1.74**	**1.11**	**-1.74**	**-14.8**	**-17.3**

==

Design an ideal band Reject filter with a frequency response Find the values of *h(n)* for N=11 using rectangular window.

$$H_d(e^{j\omega}) = \begin{cases} 1, & for|\omega| \le \frac{\pi}{3} \; and \; |\omega| \le \frac{2\pi}{3} \\ 0 & otherwise \end{cases}$$

(May/June-14)(Nov/Dec-12)(Nov/Dec-2011)

Solution:

Given:

$$H_d(e^{j\omega}) = \begin{cases} 1, & for|\omega| \le \frac{\pi}{3} \; and \; |\omega| \le \frac{2\pi}{3} \\ 0 & otherwise \end{cases}$$

$$\omega_{C1} = \frac{\pi}{3} \; and \; \omega_{C2} = \frac{2\pi}{3}$$

Step 1: Filter coefficients are,

$$h_d(n) = \frac{1}{\pi(n-\alpha)}\left[\sin \omega_{C1}(n-\alpha) - \sin \omega_{C2}(n-\alpha) + \sin(n-\alpha)\pi\right]$$

$$= \frac{1}{\pi n}\left[\sin\frac{\pi n}{3} - \sin\frac{2\pi n}{3} + \sin n\pi\right]$$

For n = 0;

$$h_d(0) = 0.667$$

$$h_d(1) = h_d(-1) = \frac{\sin\frac{\pi n}{3} - \sin\frac{2\pi n}{3} + \sin n\pi}{\pi(1)} = 0$$

$$h_d(2) = h_d(-2) = \frac{\sin\frac{\pi n}{3} - \sin\frac{2\pi n}{3} + \sin n\pi}{\pi(2)} = 0.2757$$

$$h_d(3) = h_d(-3) = \frac{\sin\frac{\pi n}{3} - \sin\frac{2\pi n}{3} + \sin n\pi}{\pi(3)} = 0$$

$$h_d(4) = h_d(-4) = \frac{\sin\frac{\pi n}{3} - \sin\frac{2\pi n}{3} + \sin n\pi}{\pi(4)} = -0.1378$$

$$h_d(5) = h_d(-5) = \frac{\sin\frac{\pi n}{3} - \sin\frac{2\pi n}{3} + \sin n\pi}{\pi(5)} = 0$$

Step 2: Using rectangular window

$$w_R(n) = \begin{cases} 1 & for -\left(\frac{M-1}{2}\right) \le n \le \left(\frac{M-1}{2}\right) \\ 0 & for\, otherwise. \end{cases}$$

$$= 1 \quad for -2 \le n \le 2$$

$$w_R(0) = w_R(1) = w_R(2) = w_R(3) = w_R(4) = w_R(5) = 1$$

Step 3: Filter coefficients using rectangular window

$$h(n) = w_R(n) * h_d(n); \quad -5 \le n \le 5$$
$$h(0) = w_R(0) * h_d(0) = 1 * 0.667 = 0.667$$
$$h(1) = w_R(1) * h_d(1) = 1 * 0 = 0$$
$$h(2) = w_R(2) * h_d(2) = 1 * 0.2757 = 0.2757$$
$$h(3) = w_R(3) * h_d(3) = 1 * 0 = 0$$
$$h(4) = w_R(4) * h_d(4) = 1 * -0.1378 = -0.1378$$
$$h(5) = w_R(5) * h_d(5) = 1 * 0 = 0$$

Step 4: The transfer function of the filter is

$$H(z) = h(0) + \sum_{n=1}^{\frac{M-1}{2}} h(n)\left[z^{-n} + z^{n}\right]$$
$$= 0.667 + 0.2757z^{2} + 0.2757z^{-2} - 0.1378z^{-4} - 0.1378z^{4}$$

Step 5: The transfer function of the realizable filter is

$$H'(z) = z^{-5}\left(0.667 + 0.2757z^{2} + 0.2757z^{-2} - 0.1378z^{-4} - 0.1378z^{4}\right)$$
$$= 0.667z^{-5} + 0.2757z^{-3} + 0.2757z^{-7} - 0.1378z^{-9} - 0.1378z^{-1}$$

The filter coefficients of the causal filers are

$$h(0) = h(10) = h(2) = h(8) = h(4) = h(6) = 0$$
$$h(1) = h(9) = -0.1378$$
$$h(3) = h(7) = 0.2757$$
$$h(5) = 0.667$$
$$H(e^{j\omega}) = \sum_{n=0}^{\frac{M-1}{2}} a(n)\cos\omega n$$
$$a(0) = h\left(\frac{M-1}{2}\right) = h(5) = 0.667$$
$$a(n) = 2h\left(\frac{M-1}{2} - n\right) = 2h(5-n)$$
$$a(1) = 2h(5-1) = 2h(4) = 0$$
$$a(2) = 2h(5-2) = 2h(3) = 0.5514$$
$$a(3) = 2h(5-3) = 2h(2) = 0$$
$$a(4) = 2h(5-4) = 2h(1) = -0.2756$$
$$a(5) = 2h(5-5) = 2h(0) = 0$$
$$\bar{H}(e^{j\omega}) = 0.667 + 0.5514\cos 2\omega - 0.2756\cos 4\omega$$

ω *(in degrees)*	**0**	**30**	**60**	**90**	**120**	**150**	**180**
$\bar{H}(e^{j\omega})$	**0.9428**	**1.08**	**0.526**	**16**	**0.529**	**1.08**	**0.9428**
$\left\|H(e^{j\omega})\right\|_{dB}$	**-0.5**	**0.67**	**-5.53**	**-15.9**	**-5.53**	**0.67**	**-0.5**

==

Design a high pass filter using window, with a cut-off frequency of 1.2 radians/sec and N=9. [Nov/Dec-2016]

Solution:

Given

$\Omega_c = 1.2 radians / \sec$

$if \;\; T = 1 \sec$

$\omega_c = \Omega_c T = 1.2 \; radians$

The impulse response of a high pass filter with a cut off frequency ω_c is

$$h_d(\text{n}) = \frac{-\sin \omega_c n}{\pi} \quad |n| \succ 0$$

$$1 - \frac{\omega_c}{\pi} \qquad \text{for n=0}$$

$$\omega_c = 1.2$$

$$h_d(0) = 1 - \frac{1.2}{\pi} = 0.618$$

$$h_d(-1) = h_d(1) = \frac{-\sin 1.2}{\pi} = -0.2966$$

$$h_d(-2) = h_d(2) = \frac{-\sin 2.4}{2\pi} = -0.1075$$

$$h_d(-3) = h_d(3) = \frac{-\sin 3.6}{3\pi} = 0.0469$$

$$h_d(-4) = h_d(4) = \frac{-\sin 4.8}{4\pi} = 0.0719$$

Hamming window for $-4 \le n \le 4 \; is$

$$w_H(\text{n}) = 0.54 + 0.46 \cos \; \frac{2\pi n}{8} \quad \text{for -4} \le \text{n} \le 4$$

$$=0.54+0.46\cos\frac{\pi n}{4}$$

$$w_H(0) = 1$$

$$w_H(-1) = w_H(1) = 0.865$$

$$w_H(-2) = w_H(2) = 0.54$$

$$w_H(-3) = w_H(3) = 0.215$$

$$w_H(-4) = w_H(4) = 0.08$$

$$\text{h(n)} = \text{h}_d\text{(n)}\,\text{w}_H\text{(n)}$$

$$h(0) = (0.618)(1) = 0.618$$

$$h(-1) = h(1) = (-0.2966)(0.865) = -0.256$$

$$h(-2) = h(2) = -0.058$$

$$h(-3) = h(3) = 0.01$$

$$h(-4) = h(4) = 0.0057$$

The casual filter coefficients are

$$h(0) = (0.618)(1) = 0.618$$

$$h(-1) = h(1) = (-0.2966)(0.865) = -0.256$$

$$h(-2) = h(2) = -0.058$$

$$h(-3) = h(3) = 0.01$$

$$h(-4) = h(4) = 0.0057$$

==

H.W: Design an ideal differentiator with frequency response $H(e^{j\omega}) = j\omega; -\pi \le \omega \le \pi$ using hamming window with N=8. (April/May-15)

==

Frequency Sampling Method:(April/May-15)

Discuss the design procedure of FIR filters using frequency sampling method. [May/June-2013]

Generally, FIR filter can be specified by giving impulse response coefficients *h(n)* (or) DFT coefficients H(k).

$$h(n) = \frac{1}{N}\sum_{k=0}^{N-1} H(k) e^{\frac{j2\pi kn}{N}} \text{ ---------------------------------> (1)}$$

$$H(k) = \sum_{n=0}^{N-1} h(n) e^{-\frac{j2\pi kn}{N}} \text{ -------------------------------------> (2)}$$

$$H(k) = DFT\ samples$$

$$H(k) = H(z)\Big|_{z=e^{\frac{j2\pi kn}{N}}} \text{ --> (3)}$$

and

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n} \text{ ---------------------------------------> (4)}$$

Put (1) in (4)

$$H(z) = \sum_{n=0}^{N-1}\left[\frac{1}{N}\sum_{k=0}^{N-1} H(k) e^{\frac{j2\pi kn}{N}}\right] z^{-n}$$

If $z = e^{j\omega}$

$$H(e^{j\omega}) = H(z)$$

$$\omega_k = \left(\frac{2\pi k}{N}\right) \qquad where\ \omega_k \text{ --> } sampling\ \ frequency.$$

General steps to design FIR filter using frequency sampling method [type-I design]:

Step 1: Draw the filter graph, as in FIR design using window function.

Step 2: Draw the unit circle and mark the points, if k-0,1…N-1.

$$\theta_k = \frac{360^o}{N} * k$$

$$if\ \ k = 1; \qquad \theta_1 = \frac{360^0}{N} * 1$$

Step 3: To find H(k), replace $\omega\ by\ \frac{2\pi k}{N}$ is the given equation.

$$H_d(e^{j\omega}) = \begin{cases} e^{-j\alpha\omega} & , \quad 0 \le \omega \le \omega_C \\ 0 & for\ otherwise \end{cases}$$

(Or)

$$H(k) = \begin{cases} e^{-\frac{j(N-1)\pi k}{N}} & , for\ k = 0\ pass\ band\ values \\ 0 & for\ k = stop\ band\ values. \end{cases}$$

$$Note: In\ type-II design\ replace\ \omega = \frac{2\pi}{N}\left(k + \frac{1}{2}\right)$$

Step 4: Find h(n)

$If\ 'N' = odd$

$$h(n) = \frac{1}{N}\left\{H(0) + \sum_{k=1}^{\frac{N-1}{2}} 2\,\mathrm{Re}\left(H(k)e^{\frac{j2\pi kn}{N}}\right)\right\} \qquad \therefore H(0) = 1$$

$If\ 'N' = Even:$

$$h(n) = \frac{1}{N}\left\{H(0) + \sum_{k=1}^{\frac{N}{2}-1} 2\,\mathrm{Re}\left(H(k)e^{\frac{j2\pi kn}{N}}\right)\right\}$$

Step 5: Find H(z):

$$H(z) = \sum_{n=0}^{N-1} h(n)z^{-n}$$

Design LPF which has the following specifications, N=7 using frequency sampling Technique. [Nov/Dec-2016][Nov/Dec-15]

$$H_d(e^{j\omega}) = \begin{cases} e^{-j3\omega} & ,0 \le \omega \le \frac{\pi}{2} \\ 0 & ,\frac{\pi}{2} \le \omega \le \pi. \end{cases}$$

Solution:

Given: $$H_d(e^{j\omega}) = \begin{cases} e^{-j3\omega} & ,0 \le \omega \le \frac{\pi}{2} \\ 0 & ,\frac{\pi}{2} \le \omega \le \pi. \end{cases}$$

Step 1: From a unit circle, mark points from

$$\theta_k = \frac{360^o}{N} * k$$

$$\theta_1 = 51.42^0$$

$$\theta_2 = 102.8^0$$

$$\theta_3 = 154.26^0$$

$$\theta 4 = 205.68^0$$

$$\theta_5 = 257.1^0$$

$$\theta_6 = 308.52^0$$

Step 2:

$$H(k) = \begin{cases} e^{-j\left(\frac{N-1}{N}\right)\pi k} & for\ k = 0,1,6 \\ 0 & for\ otherwise \end{cases}$$

N=7.

$$H(k) = \begin{cases} e^{-j\frac{6\pi k}{7}} & for\ k = 0,1,6 \\ 0 & for\ k = 2,3,4,5 \end{cases}$$

$$h(n) = \frac{1}{N}\left\{H(0) + \sum_{k=1}^{\frac{N-1}{2}} 2\,\mathrm{Re}\left(H(k)e^{\frac{j2\pi kn}{N}}\right)\right\} \qquad \therefore H(0) = 1$$

$$= \frac{1}{7}\left\{1 + \sum_{k=1}^{3} 2\,\mathrm{Re}\left(e^{-j\frac{6\pi k}{7}}.e^{\frac{j2\pi kn}{7}}\right)\right\}$$

$$= \frac{1}{7}\left\{1 + 2\sum_{k=1}^{3} \mathrm{Re} \quad .e^{\frac{j\pi k(2n-6)}{7}}\right\}$$

$$h(n) = \frac{1}{7}\left\{1 + 2\sum_{k=1}^{3} \cos\left(\frac{\pi k(2n-6)}{7}\right)\right\}$$

$$h(n) = \frac{1}{7}\left\{1 + 2\left(\cos\left(\frac{\pi(2n-6)}{7}\right)\right)\right\} \quad for\ k = 1.$$

$$h(0) = -0.114; h(1) = 0.079; h(2) = 0.321; h(3) = 0.4286; h(4) = h(2); h(5) = h(1); h(6) = h(0)$$

==

Find coefficient of LP FIR with N=15 and it has symmetric unit sample response. It satisfies the following condition.

$$H\left(\frac{2\pi k}{15}\right) = \begin{cases} 1 & ,k = 0,1,2,3 \\ 0.4 & ,k = 4 \\ 0 & ,k = 5,6,7. \end{cases}$$

Solution:

Step 1:

$$H(k) = \begin{cases} 1.e^{-j\left(\frac{N-1}{N}\right)\pi k} & ,k = 0,1,2,3 \\ 0.4e^{-j\left(\frac{N-1}{N}\right)\pi k} & ,k = 4 \\ 0 & ,k = 5,6,7 \end{cases}$$

$$H(k) = \begin{cases} 1.e^{-j\frac{14\pi k}{15}} & ,k = 0,1,2,3 \\ 0.4e^{-j\frac{14\pi k}{15}} & ,k = 4 \\ 0 & ,k = 5,6,7 \end{cases}$$

Step 2:

$$h(n) = \frac{1}{N}\left\{H(0) + \sum_{k=1}^{\frac{N-1}{2}} 2\,\mathrm{Re}\left(H(k)e^{\frac{j2\pi kn}{N}}\right)\right\} \qquad \therefore H(0) = 1$$

$$h(n) = \frac{1}{15}\left\{1 + \sum_{k=1}^{7} 2\,\mathrm{Re}\left(\left(e^{-\frac{j14\pi k}{15}} + 0.4e^{-\frac{j14\pi k}{15}}\right)e^{\frac{j2\pi kn}{15}}\right)\right\}$$

$$= \frac{1}{15}\left\{1 + 2\,\mathrm{Re}\left[\sum_{k=1}^{3} e^{\frac{j2\pi k(n-7)}{15}}\right] + 2(0.4)\sum_{k=4} e^{\frac{j2\pi k}{15}(n-7)}\right\}$$

$$= \frac{1}{15}\left\{1 + 2\left[\cos\frac{2\pi(n-7)}{15} + \cos\frac{4\pi(n-7)}{15} + \cos\frac{6\pi(n-7)}{15}\right] + 0.8\cos\left(\frac{8\pi(n-7)}{15}\right)\right\}$$

$h(0) = -0.0141; h(1) = -0.0195; h(2) = 0.04; h(3) = 0.0122$

$h(4) = -0.0913; h(5) = -0.01809; h(6) = 0.313; h(7) = 0.52.$

==

Determine the coefficients {h (n)} of a linear phase FIR filter of length M =15 has a symmetric unit sample response and a frequency response that satisfies the condition $H_r\left(\frac{2\pi k}{15}\right) = \begin{cases} 1 \ for\ k = 0,1,2,3 \\ 0 \ for\ k = 4,5,6,7 \end{cases}$ **(May/June-13) (April/May-11)(Nov/Dec-09)**

Solution:

$|\overline{H}(\mathrm{k})| = 1 \qquad \text{for } 0 \le \mathrm{k} \le 3 \text{ and } 12 \le \mathrm{k} \le 14$

$= 0 \qquad \text{for } 4 \le \mathrm{k} \le 11$

$$\theta(\mathrm{k}) = -\left(\frac{N-1}{N}\right)\pi\,\mathrm{k}$$

$$= -14/15\ \pi\mathrm{k} \quad 0 \le \mathrm{k} \le 7$$

and

$$\theta(\mathrm{k}) = 14\pi - \frac{14\pi k}{15} \quad \text{for } 8 \le \mathrm{k} \le 14$$

$$H(\mathrm{k}) = \mathrm{e}^{-j14\pi k/15} \qquad \text{for k=0,1,2,3}$$

$$= 0 \qquad \text{for } 4 \le \mathrm{k} \le 11$$

$$= \mathrm{e}^{-j14\pi(\mathrm{k}-15)/15} \qquad \text{for } 12 \le \mathrm{k} \le 14$$

$$h(\mathrm{n}) = \frac{1}{n}\left[H(0) + 2\sum_{k=1}^{\frac{N-1}{2}} \mathrm{Re}(H(\mathrm{k})\mathrm{e}^{j2\pi nk/15})\right]$$

$$= \frac{1}{5}\left[1 + 2\sum_{k=1}^{7} \mathrm{Re}(e^{-j14\pi k/15}e^{j2\pi nk/15}\right]$$

$$= \frac{1}{15}\left[1 + 2\sum_{k=1}^{3} \cos\frac{2\pi k(7-\mathrm{n})}{15}\right]$$

$$= \frac{1}{15}\left[1 + 2\cos\frac{2\pi(7-\mathrm{n})}{15} + 2\cos\frac{4\pi(7-\mathrm{n})}{15} + 2\cos\frac{6\pi(7-\mathrm{n})}{15}\right]$$

` $h(0) = \mathrm{h}(14) = -0.05;\ h(1) = \mathrm{h}(3) = 0.041;\ h(4) = \mathrm{h}(10) = -0.1078$

$h(2) = \mathrm{h}(12) = 0.0666;\ \ h(3) = \mathrm{h}(11) = -0.0365;\ h(5) = \mathrm{h}(9) = 0.034$

$h(6) = \mathrm{h}(8) = 0.3188;\ \ h(7) = 0.466$.

Using frequency sampling method, design BPF with the following specifications. [May/June-2016]
Sampling frequency F=8000Hz
Cut off frequencies f_{c1}=1000Hz
Cut off frequencies f_{c2}=3000Hz Determine the filter coefficients for N=7.

Solution:

$$\omega_{c_1} = 2\pi f_{c_1} T = \frac{2\pi f_{c_1}}{F} = \frac{2\pi(1000)}{8000} = \frac{\pi}{4}$$

$$\omega_{c_2} = 2\pi f_{c_2} T = \frac{2\pi f_{c_2}}{F} = \frac{2\pi(3000)}{8000} = \frac{3\pi}{4}$$

$$H(K) = H_d\left(e^{j\omega}\right)\Big|_{\omega=\frac{2\pi}{7}k} \qquad k=0,1,\ldots 6$$

$$|H(k)| = \begin{cases} 0 & for\, k = 0,3 \\ 1 & for\, k = 1,2 \end{cases}$$

$$\theta(k) = -\left(\frac{N-1}{N}\right)\pi \quad for \quad 0 \le k \le \frac{N-1}{2}$$

$$= -\frac{6}{7}\pi k \quad for \quad 0 \le k \le 3$$

$$|H(k)| = \begin{cases} 0 & for\, k = 0,3 \\ e^{\frac{-j6\pi k}{7}} & for\, k = 1,2 \end{cases}$$

The filter coefficients are given by

$$h(n) = \frac{1}{N}\left[H(0) + \sum_{k=1}^{\frac{N-1}{2}} 2.\mathrm{Re}\left[H(k).e^{\frac{j2\pi kn}{N}}\right]\right]$$

$$= \frac{1}{7}\left[2\sum_{k=1}^{\frac{N-1}{2}} \mathrm{Re}\left[e^{\frac{j6\pi k}{7}} e^{\frac{j2\pi kn}{7}}\right]\right]$$

$$= \frac{1}{7}\left[2\sum_{k=1}^{\frac{N-1}{2}} \cos\frac{2\pi k}{7}((3-n))\right]$$

$$= \frac{2}{7}\left[\cos\frac{2\pi}{7}(3-n) + \cos\frac{4\pi}{7}(3-n)\right]$$

$h(0) = h(6) = -0.07928$.

$h(1) = h(5) = -0.321$

$h(2) = h(4) = -0.11456$

$h(3) = 0.57$

****

H.W:1. Design a linear phase FIR high pass filter using hamming window, with a cutoff frequency, ω_c=0.8π rad/sample and N=7.

2. Design a FIR low pass filter with cutoff frequency of 1KHz and sampling frequency of 4 kHz with 11 samples using Fourier series method. Determine the frequency responses and verify the design by sketching the magnitude responses

3. Using a rectangular window technique design a low pass filter with pass band gain of unity, cutoff frequency of 1000Hz and working at a sampling frequency of 5kHz.the length of the impulse response should be 7.

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UNIT - IV FINITE WORD LENGTH EFFECTS IN DIGITAL FILTER

Fixed point and floating point number representation - ADC - quantization - truncation and rounding - quantization noise - input / output quantization - coefficient quantization error - product quantization error - overflow error - limit cycle oscillations due to product quantization and summation - scaling to prevent overflow.

Finite Word length Effects:

- In the design of FIR Filters, The filter coefficients are determined by the system transfer functions. These filter co-efficient are quantized/truncated while implementing DSP System because of finite length registers.
- Only Finite numbers of bits are used to perform arithmetic operations. Typical word length is 16 bits, 24 bits, 32 bits etc.
- This finite word length introduces an error which can affect the performance of the DSP system.
- The main errors are
  1. Input quantization error
  2. Co-efficient quantization error
  3. Overflow & round off error (Product Quantization error)
- The effect of error introduced by a signal process depend upon number of factors including the
  1. Type of arithmetic
  2. Quality of input signal
  3. Type of algorithm implemented

1. Input quantization error

- The conversion of continuous-time input signal into digital value produces an error which is known as input quantization error.
- This error arises due to the representation of the input signal by a fixed number of digits in A/D conversion process.

2. Co-efficient quantization error

- The filter coefficients are compared to infinite precision. If they are quantized the frequency response of the resulting filter may differ from the desired frequency response.
  i.e poles of the desired filter may change leading to instability.

3. **Product Quantization error**

- It arises at the output of the multiplier
- When a ‘b’ bit data is multiplied with another ‘b’ bit coefficient the product (‘2b’ bits) should be stored in ‘b’ bits register. The multiplier Output must be rounded or truncated to ‘b’ bits. This known as overflow and round off error.

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Types of number representation:

There are two common forms that are used to represent the numbers in a digital or any other digital hardware.

1. Fixed point representation
2. Floating point representation

*** Explain the various formulas of the fixed point representation of binary numbers.**

1. Fixed point representation

- In the fixed point arithmetic, the position of the binary point is fixed. The bit to the right represents the fractional part of the number and to those to the left represents the integer part.

- For example, the binary number 01.1100 has the value 1.75 in decimal.

  $(0*2^{1)} + (1*2^0) + (1*2^{-1}) + (1*2^{-2}) + (0*2^{-3}) = 1.75$

  In general, we can represent the fixed point number ‘N’ to any desired accuracy by the series

$$N = \sum_{i=n_i}^{n_2} C_i r^i$$

  Where, r is called as radix.
- If r=10, the representation is known as decimal representation having numbers from 0 to 9. In this representation the number

$$30.285 = \sum_{i=-3}^{1_2} C_i 10^i$$

$$= (3*10^1)+ (0*10^0)+ (2*10^{-1})+(8*10^{-2})+(5*10^{-3})$$

- If r=2, the representation is known as binary representation with two numbers 0 to 1.
- For example, the binary number

$$110.010 = (1*2^2) + (1*2^1) + (0*2^0) + (0*2^{-1}) + (1*2^{-2}) + (0*2^{-3}) = 6.25$$

Examples:

Convert the decimal number 30.275 to binary form

2	**30**	
2	**15**	**--0**
2	**7**	**--1**
2	**3**	**--1**
	1	**--1**

0.275 * 2	→0.55	→0
0.55 * 2	→1.10	→1
0.10 * 2	→0.20	→0
0.20 * 2	→0.40	→0
0.40 * 2	→0.80	→0
0.80 * 2	→1.60	→1
0.60 * 2	→1.20	→1
0.20 * 2	→0.40	→0

$(30.275)_{10}$ = $(11110.01000110)_2$

In fixed point arithmetic =, the negative numbers are represented by 3 forms.

1. Sign-magnitude form
2. One’s complement form
3. Two’s complement form

1.1 Sign-magnitude form:

- Here an additional bit called sign bit is added as MSB.
  - If this bit is zero → It is a positive number
  - If this bit is one → It is a positive number
- For example
  - 1.75 is represented as 01.110000.
  - -1.75 is represented as 11.110000

1.2 One’s complement form:

- Here the positive number is represented same as that in sign magnitude form.
- But the negative number is obtained by complementing all the bits of the positive number
- For eg: the decimal number -0.875 can be represented as
  - $(0.875)_{10}=(0.111000)_2$
  - $(-0.875)_{10}=(1.000111)_2$

0.111000
↓ ↓↓↓↓↓↓ (Complement each bit)
1.000111

1.3 Two's complement form:

- Here the positive numbers are represented as same in sign magnitude and one's complement form.
- The negative numbers are obtained by complementing all the bits of the positive number and adding one to the least significant bit

  $(0.875)_{10}=(0.111000)_2$

  ↓ ↓↓↓↓↓↓ (Complement each bit)

  1.000111

  \+ 1

  1.001000

  $(-0.875)_{10}=(1.001000)_2$

Examples:

- **Find the sign magnitude, 1's complement, 2's complement for the given numbers.**

  1. $-\frac{7}{32}$

  2. $-\frac{7}{8}$

  3. $+\frac{7}{8}$

1. $-\frac{7}{32}$

0.21875 * 2 ➔0.43750 ➔0

0.43750 * 2 ➔0.87500 ➔0

0.87500* 2 ➔ 1.750000 ➔1

0.75* 2 ➔1.50 ➔1

0.50* 2 ➔1.00 ➔1

$-\frac{7}{32}=(-0.21875)_{10} =(1.00111)_2$

Sign magnitude form = 1.00111

1's complement form = 1.11000

2's complement form = 1.11001

2. $-\frac{7}{8}$

0.875 * 2 ➔1.75 ➔1

0.750 * 2 ➔1.500 ➔1

0.500 * 2 ➔1.000 ➔1

$-\frac{7}{8}=(-0.875)_{10} =(0.111)_2$

Sign magnitude form = 0.111

1's complement form = 1.000

2's complement form = 1.001

3. $+\frac{7}{8}$

Sign magnitude form = 0.111

1's complement form = 0.111

2's complement form = 0.111

Addition of two fixed point numbers:

- **Add $(0.5)_{10} + (0.125)_{10}$**

$(0.5)_{10}$ = $(0.100)_2$
$(0.125)_{10}$ = $(0.001)_2$
$(0.101)_2 = (0.625)_{10}$

- Addition of two fixed point numbers causes an overflow.
  For example
  $(0.100)_2$
  $(0.101)_2$
  $(1.001)_2 = (-0.125)_{10}$ in sign magnitude form

Subtraction of two fixed point numbers:

- Subtraction of two numbers can be easily performed easily by using two's complement representation.
- **Subtract 0.25 from 0.5**

0.25 * 2 → 0.50	→ 0	Sign magnitude form =	$(0.010)_2$
0.50 * 2 → 1.00	→ 1	1's complement form =	$(1.101)_2$
0.00 * 2 → 0.00	→ 0	2's complement form =	$(1.110)_2$

$(0.5)_{10}$ = $(0.100)_2$
$-(0.25)_{10}$ = $(1.110)_2$ → Two's complement of -0.25
$(10.010)_2$

Here the carry is generated after the addition. Neglect the carry bit to get the result in decimal.

$(0.010)_2 = (0.25)_{10}$

- **Subtract 0.5 from 0.25**

0.5 * 2 → 1.00	→ 1	Sign magnitude form =	$(0.100)_2$
0.00 * 2 → 0.00	→ 0	1's complement form =	$(1.011)_2$
0.00 * 2 → 0.00	→ 0	2's complement form =	$(1.100)_2$

$(0.25)_{10}$ = $(0.010)_2$
$-(0.5)_{10}$ = $(1.100)_2$
$(1.110)_2$

Here the carry is not generated after the addition. So the result is negative.

Multiplication in fixed point arithmetic:

- Here the sign magnitude components are separated.
- The magnitudes of the numbers are multiplied. Then the sign of the product is determined and applied to the result.
- In the fixed point arithmetic, multiplication of two fractions results in a fraction.
- For multiplications with fractions, overflow can never occur.
- Eg:
  $(0.1001)_2 * (0.0011)_2$ = $(0.00011011)_2$

2. Floating point representation

- Here, a number 'x' is represented by
  $X=M.r^e$
  Where, M → Mantissa which requires a sign bit for representing positive number and negative numbers.
  R → base (or) radix
  e → exponent which require an additional and it may be either positive or negative.
  - For eg, 278 can be represented in floating point representation.

$$278 = \frac{278 \text{ X } 1000}{1000} = 0.278*10_3$$

0.278 → Mantissa (M)
10 → base (or) radix (r)
3 → exponents (e)

- Similarly, to represent a binary floating point number
  $X=M.2^e$ in which the fractional part of a number should fall (or) lie in the range of 1/2 to 1.

$$5 = \frac{5 \ X \ 8}{8} \qquad 0.625 \ X2^3$$

Mantissa (M)	=	0.625
Base (or) radix (r)	=	2
Exponent (e)	=	3

- Some decimal numbers and their floating point representations are given below:

4.5	→	$0.5625 \ X \ 2^3$	$=0.1001 \ X \ 2^{011}$
1.5	→	$0.75 \ X \ 2^1$	$=0.1100 \ X \ 2^{001}$
6.5	→	$0.8125 \ X \ 2^3$	$=0.1100 \ X \ 2^{011}$
0.625	→	$0.625 \ X \ 2^0$	$=0.1010 \ X \ 2^{000}$

- Negative floating point numbers are generally represented by considering the mantissa as a fixed point number. The sign of the floating point number is obtained from the first bit of mantissa.
- To represent floating point in multiplication

  Consider $X_1 = M_1 r^{e_1}$

  $X_2 = M_2 r^{e_2}$

  $X_1 X_2 = (M_1 * M_2) r^{(e_1+e_2)}$

Example

Given $X_1 = 3.5*10^{-12}$, $X_2 = 4.75*10^6$. Find the product X_1X_2

$X=(3.5 \ X \ 4.75) \ 10^{(-12+6)}$

$= (16.625)10^{-6}$ → in decimal

In binary: $(1.5)_{10} \ X \ (1.25)_{10}$ = $(2^1 0.75) \ X \ (2^1 0.625)$

= $2^{001} \ X \ 0.1100 \ X \ 2^{001} \ X \ 0.1010$

= $2^{010} \ X \ 0.01111$

Addition and subtraction:

- Here the exponent of a smaller number is adjusted until it matches the exponent of a larger number.
- Then, the mantissa are added or subtracted
- The resulting representation is rescaled so that its mantissa lies in the range 0.5 to 1.
- Eg: **Add $(3.0)_{10}$ & $(0.125)_{10}$**

  $(3.0)_{10}$ $=2^{010} \ X \ 0.1100 = r^{e_1} \times M_1$

  $(0.125)_{10}$ $=2^{000} \ X \ 0.0010 = r^{e_2} \times M_2$

  Now adjust e_2 Such that $e_1=e_2$

  $(0.125)_{10}$ $=2^{010} \ X \ 0.0000100$

  Addition →2^{010} (0.110000 + 0.0000100) →$2^{010} \ X \ 0.110010$

  Subraction →$2^{010} \ X \ 1.001101$

Compare floating point with fixed point arithmetic.

Sl.No	Fixed point arithmetic	Floating point arithmetic
1	Fast operation	Slow operation
2	Relatively economical	More expensive because of costlier hardware
3	Small dynamic range	Increased Dynamic range
4	Round off errors occurs only for addition	Round off errors can occur with addition and multiplication
5	Overflow occur in addition	Overflow does not arise
6	Used in small computers	Used in large general purpose computers.

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Quantization:

***Discuss the various methods of quantization.**

***Derive the expression for rounding and truncation errors**

*** Discuss in detail about Quantization error that occurs due to finite word length of registers.**

The common methods of quantization are

1. Truncation
2. Rounding

1. Truncation

- The abrupt termination of given number having a large string of bits (or)
- Truncation is a process of discarding all bits less significant than the LSB that is retained.
- Suppose if we truncate the following binary number from 8 bits to 4 bits, we obtain
    - 0.00110011 to 0.0011
      (8 bits) (4 bits)
    - 1.01001001 to 1.0100
      (8 bits) (4 bits)
- When we truncate the number, the signal value is approximated by the highest quantization level that is not greater than the signal.

2. Rounding (or) Round off

- Rounding is the process of reducing the size of a binary number to finite word size of 'b' bits such that the rounded b-bit number is closest to the original unquantised number.

Error Due to truncation and rounding:

- While storing (or) computation on a number we face registers length problems. Hence given number is quantized to truncation (or) round off.
  i.e. Number of bits in the original number is reduced register length.

Truncation error in sign magnitude form:

- Consider a 5 bit number which has value of
  $0.11001_2 \rightarrow (0.7815)_{10}$
- This 5 bit number is truncated to a 4 bit number
  $0.1100_2 \rightarrow (0.75)_{10}$
  i.e. 5 bit number →0.11001 has 'l' bits
  4 bit number →0.1100 has 'b' bits
- Truncation error, e_t = 0.1100 – 0.11001
  = -0.00001 → $(-0.03125)_{10}$
- Here original length is 'l' bits. (l=5). The truncated length is 'b' bits.
- The truncation error, $e_t = 2^{-b}-2^{-l}$
  $= -(2^{-l}-2^{-b})$
  $e_t = -(2^{-5}-2^{-4}) = -2^{-1}$
- The truncation error for a positive number is
  $-\left(2^{-b}-2^{-l}\right) \le e_t \le 0$ → Non causal
- The truncation error for a negative number is
  $0 \le e_t \le \left(2^{-b}-2^{-l}\right)$ → Causal

Truncation error in two's complement:

- For a positive number, the truncation results in a smaller number and hence remains same as in the case of sign magnitude form.
- For a negative number, the truncation produces negative error in two's complement

$$-\left(2^{-b}-2^{-l}\right) \le e_t \le \left(2^{-b}-2^{-l}\right)$$

Round off error (Error due to rounding):

- Let us consider a number with original length as '5' bits and round off length as '4' bits.

$$0.11001 \xrightarrow{\text{Round off to}} 0.1101$$

- Now error due to rounding $e_r = \frac{2^{-b} - 2^{-l}}{2}$

  Where b→Number of bits to the right of binary point after rounding
  L→Number of bits to the right of binary point before rounding

- Rounding off error for positive Number:

$$-\frac{2^{-b} - 2^{-l}}{2} \le e_r \le 0$$

- Rounding off error for negative Number:

$$0 \le e_r \le \frac{2^{-b} - 2^{-l}}{2}$$

- For two's complement

$$-\frac{2^{-b} - 2^{-l}}{2} \le e_r \le \frac{2^{-b} - 2^{-l}}{2}$$

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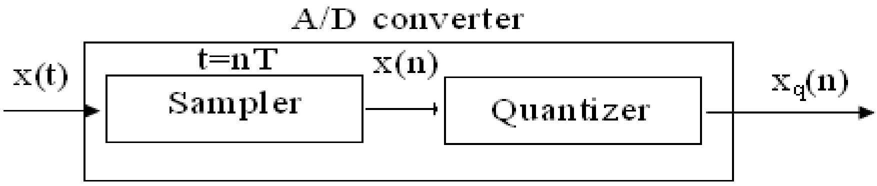
Quantization Noise:

***Derive the expression for signal to quantization noise ratio**

***What is called Quantization Noise? Derive the expression for quantization noise power.**



- The analog signal is converted into digital signal by ADC
- At first, the signal x(t) is sampled at regular intervals t=nT, where n=0,1,2… to create sequence x(n). This is done by a sampler.
- Then the numeric equivalent of each sample x(n) is expressed by a finite number of bits giving the sequence $x_q(n)$
- The difference signal $e(n)= x_q(n)- x(n)$ is called quantization noise (or) A/D conversion noise.
- Let us assume a sinusoidal signal varying between +1 & -1 having a dynamic range 2
- ADC employs (b+1) bits including sign bit. In this case, the number of levels available for quantizing x(n) is 2^{b+1}.
- The interval between the successive levels is

  $$q = \frac{2}{2^{b+1}} = 2^{-b}$$

  Where q → quantization step size
  If b=3 bits, then $q=2^{-3}=0.125$

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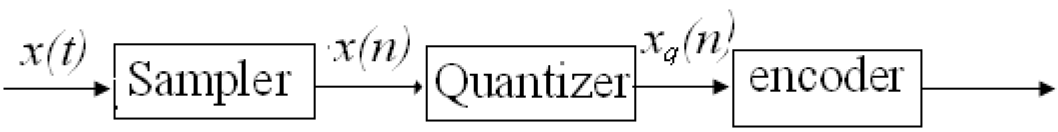
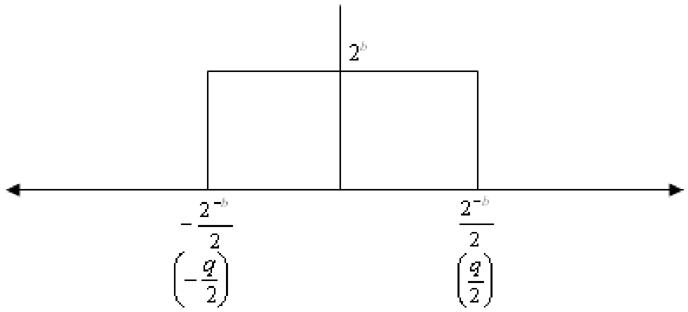
Quantization Noise power:

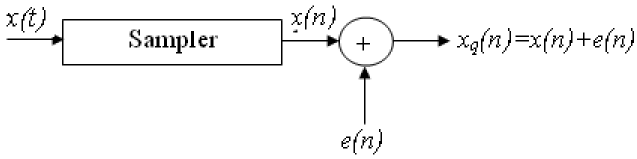
<u>Input Quantization error:</u>

***Derive the equation for quantization noise power (or) Steady state Input Noise Power.**



Probability density function for round off error in A/D conversion is





If rounding is used for quantization, which is bounded by $-\frac{q}{2} \le e(n) \le \frac{q}{2}$, then the error lies between $-\frac{q}{2}$ to $\frac{q}{2}$ with equal probability, where q→ quantization step size.

<u>Properties of analog to digital conversion error, e(n):</u>

1. The error sequence e(n) is a sample sequence of a stationary random process.
2. The error sequence is uncorrelated with x(n) and other signals in the system.
3. The error is a white noise process with uniform amplitude probability distribution over the range of quantization error.

The variance of e(n) is given by

$$\sigma_e^2 = E\left[e^2(n)\right] - E^2\left[e(n)\right] \text{ ---------------------------->(1)}$$

Where $E\left[e^2(n)\right]$→Average of $e^2(n)$

$E\left[e(n)\right]$→Mean value of e(n).

For rounding, e(n) lies between $-\frac{q}{2}$ and $\frac{q}{2}$ with equal probability

$$E\left[e^2(n)\right] = \int_{-\infty}^{\infty} e^2(n)p(e)de \text{ ---------------------------->(2)}$$

$$p(e) = \frac{1}{q},\ -\frac{q}{2} \le e(n) \le \frac{q}{2} \text{ ---------------------------->(3)}$$

Substituting (3) in (2)

$$E\left[e^2(n)\right] = \int_{-\frac{q}{2}}^{\frac{q}{2}} e^2(n)\frac{1}{q}de$$

$$E\left[e^2(n)\right] = \frac{1}{q}\int_{-\frac{q}{2}}^{\frac{q}{2}} e^2(n)de \text{ ------------------------------>(4)}$$

$$E\left[e(n)\right] = 0$$

$$E^2\left[e(n)\right] = 0 \text{ -->(5)}$$

Substituting (4) and (5) in (1)

$$\sigma_e^2 = \frac{1}{q}\int_{-\frac{q}{2}}^{\frac{q}{2}} e^2(n)de - 0$$

$$= \frac{1}{q}\left[\frac{e^3}{3}\right]_{-\frac{q}{2}}^{\frac{q}{2}}$$

$$= \frac{1}{3q}\left[\left(\frac{q}{2}\right)^3 - \left(-\frac{q}{2}\right)^3\right]$$

$$= \frac{1}{3q}\left[\left(\frac{q^3}{8}\right) - \left(-\frac{q^3}{8}\right)\right]$$

$$= \frac{1}{3q}\left[\left(\frac{q^3}{8}\right) + \left(\frac{q^3}{8}\right)\right]$$

$$= \frac{1}{3q}\left[\frac{2q^3}{8}\right]$$

$$\sigma_e^2 = \frac{q^2}{12} \text{--->(6)}$$

In general, $\frac{1}{2^b} = 2^{-b} = q$ --->(7)

$$\sigma_e^2 = \frac{\left(2^{-b}\right)^2}{12}$$

$$\sigma_e^2 = \frac{2^{-2b}}{12} \text{ -->(8)}$$

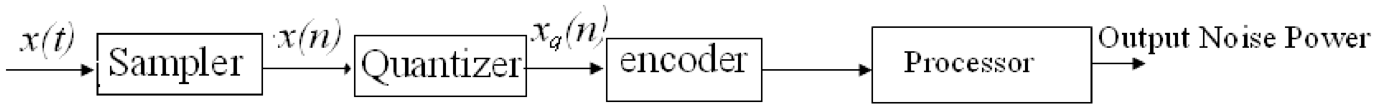
Equation (8) is known as the steady state noise power due to input quantization.

$q = \frac{R}{2^b}$ → in two's complement representation.

$q = \frac{R}{2^b - 1}$ → in sign magnitude (or) one's complement representation.

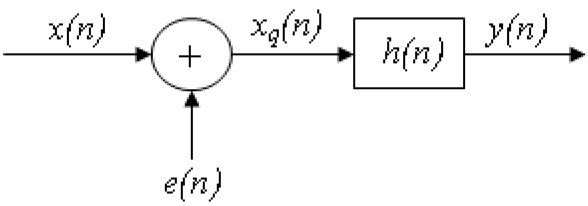
R → Range of analog signal to be quantized.

Steady state Output Noise power:



After quantization, we have noise power σ_e^2 as input noise power. Therefore, Output noise power of system is given by

$$\sigma_{eo}^2 = \sigma_e^2\left[\sum_{n=0}^{\infty} h^2(n)\right] \text{ ------------------------------------>(9)}$$



where h(n) → impulse response of the system.

Let error E(n) be output noise power due to quantization

Error $E(n) = e(n) * h(n)$

$$= \sum_{k=0}^{\infty} h(n) e(n-k)$$

The variance of error E(n) is called output noise power, σ_e^2.

By using Parseval's theorem,

$$\sigma_{eo}^2 = \sigma_e^2 \sum_{n=0}^{\infty} h^2(n)$$

$$= \sigma_e^2 \frac{1}{2\pi j} \oint H(Z) H(Z^{-1}) \frac{dZ}{z}$$

Where the closed contour integration is evaluated using the method of residue by taking only the poles that lie inside the unit circle.

Z transform of h(n), $H(Z) = \sum_{n=0}^{\infty} h(n) z^{-n}$

Z transform of $h^2(n)$ = Z[$h^2(n)$] $= \sum_{n=0}^{\infty} h^2(n) z^{-n} = \sum_{n=0}^{\infty} h(n) h(n) z^{-n}$ -------------------->(10)

By Inverse Z transform, $h(n) = \frac{1}{2\pi j} \oint H(Z) Z^{n-1} dZ$ -------------------------------->(11)

Substituting (11) in (10)

$$\sum_{n=0}^{\infty} h^2(n) z^{-n} = \sum_{n=0}^{\infty} \frac{1}{2\pi j} \oint H(Z) Z^{n-1} dZ \; h(n) z^{-n}$$

$$= \frac{1}{2\pi j} \oint H(Z) \left[\sum_{n=0}^{\infty} h(n) Z^{-1} \right] dZ$$

$$\sum_{n=0}^{\infty} h^2(n) = \frac{1}{2\pi j} \oint H(Z) \left[\sum_{n=0}^{\infty} h(n) Z^{-1} \right] \frac{dZ}{Z^{-n}}$$

$$= \frac{1}{2\pi j} \oint H(Z) \left[\sum_{n=0}^{\infty} h(n) (Z^{-n})^{-1} Z^{-1} dZ \right]$$

$$\sum_{n=0}^{\infty} h^2(n) = \frac{1}{2\pi j} \oint H(Z) H(Z^{-1}) \frac{dZ}{Z}$$ ---------------------------------->(12)

Substituting (12) in (9)

$$\sigma_{eo}^2 = \sigma_e^2 \left[\frac{1}{2\pi j} \oint H(Z) H(Z^{-1}) Z^{-1} dZ \right]$$

Problem:

The output signal of an A/D converter is passed through a first order low pass filter, with transfer function given by

$H(z) = \frac{(1-a)z}{z-a}$ *for* $0 < a < 1$. **Find the steady state output noise power due to quantization at the output of the digital filter. [Nov/Dec-2015]**

Solution:

$$\sigma_e^2 = \sigma_e^2 \frac{1}{2\pi j} \oint_c H(z) H(z^{-1}) z^{-1} dz$$

$$Given \quad H(z) = \frac{(1-a)z}{(z-a)} \qquad H(z^{-1}) = \frac{(1-a)z^{-1}}{(z^{-1}-a)}$$

Substituting $H(z)$ *and* $H(z^{-1})$ *in equation (1), we have*

$$\sigma_e^2 = \frac{\sigma_e^2}{2\pi j} \oint_c \frac{(1-a)z}{(z-a)} \frac{(1-a)z^{-1}}{(z^{-1}-a)} z^{-1} dz = \frac{\sigma_e^2}{2\pi j} \oint_c \frac{(1-a)^2}{(z-a)} \frac{dz}{(z^{-1}-a)} z^{-1}$$

$$= \sigma_e^{\,2}\left[residue\ of\ H(z)\ H(z^{-1})z^{-1} at\ z = a\ + residue\ of\ H(z)\ H(z^{-1})z^{-1} at\ z = \frac{1}{a} \right]$$

$$= \sigma_e^{\,2}\left[(z-a)\frac{(1-a)^2 z^{-1}}{(z-a)(z^{-1}-a)} + 0 \right]$$

$$= \sigma_e^{\,2}\left[\frac{(1-a)^2}{(z^{-1}-a)} \right] = \sigma_e^{\,2}\left[\frac{(1-a)}{(1+a)} \right]$$

$$Where,\quad \sigma_e^{\,2} = \frac{2^{-2b}}{12}$$

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Find the steady state variance of the noise in the output due to quantization of input for the first order filter. [Apr/May'11] [Nov/Dec-2016]

$$y(n) = ay(n-1) + x(n)$$

Solution:

The impulse response for the above filter is given by $h(n) = a^n u(n)$

$$\sigma_\epsilon^2 = \sigma_e^2 \sum_{k=0}^{\alpha} h^2(n)$$

$$= \sigma_e^2 \sum_{k=0}^{\alpha} a^{2n}$$

$$= \sigma_e^2 \left[1 + a^2 + a^4 + ...\infty \right]$$

$$= \sigma_e^2 \frac{1}{1-a^2}$$

$$= \frac{2^{-2b}}{12}\left[\frac{1}{1-a^2} \right] \qquad (or)$$

Taking Z-transform on both sides we have

$$Y(z) = az^{-1}Y(z) + X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-az^{-1}} = \frac{z}{z-a}$$

$$H(z^{-1}) = \frac{z^{-1}}{z^{-1}-a}$$

We know

$$\sigma_\epsilon^2 = \sigma_e^2 \frac{1}{2\pi j} \oint_c H(z)H(z^{-1})z^{-1}dz$$

Substituting $H(z)$ *and* $H(z^{-1})$ *values in the above equation we get*

$$\sigma_\epsilon^2 = \sigma_e^2 \frac{1}{2\pi j} \oint_c \frac{z}{z-a}\frac{z^{-1}}{z^{-1}-a} z^{-1}dz$$

$$\sigma_\epsilon^2 = \sigma_e^2 \frac{1}{2\pi j} \oint_c \frac{z^{-1}}{(z-a)(z^{-1}-a)}dz$$

$$= \sigma_e^2 \left[\begin{array}{l} residue\ of\ \dfrac{z^{-1}}{(z-a)(z^{-1}-a)}\ at\ z = a \\ +residue\ of\ \dfrac{z^{-1}}{(z-a)(z^{-1}-a)}\ at\ z = 1/a \end{array} \right]$$

$$= \sigma_e^2 \left[(z-a)\frac{z^{-1}}{(z-a)(z^{-1}-a)}\bigg|_{z=a} \right]$$

$$= \sigma_e^2 \frac{a^{-1}}{a^{-1} - a} = \sigma_e^2 \frac{1}{1 - a^2}$$

****

The output of the A/D converter is applied to a digital filter with the system function

$$H(Z) = \frac{0.45Z}{Z - 0.72}$$

Find the output noise power of the digital filter, when the input signal is quantized to 7 bits.

Given:

$$H(Z) = \frac{0.45Z}{Z - 0.72}$$

Solution:

$$H(Z)H(Z^{-1})Z^{-1} = \frac{0.45Z}{Z - 0.72} \times \frac{0.45Z^{-1}}{Z^{-1} - 0.72} \times Z^{-1}$$

$$= \frac{0.45^2 Z^{-1}}{(Z - 0.72)\left(\frac{1}{Z} - 0.72\right)}$$

$$= \frac{0.2025Z^{-1}}{(Z - 0.72)\left(\frac{1 - 0.72Z}{Z}\right)}$$

$$= \frac{0.2025Z^{-1}Z}{(Z - 0.72)\left(Z - \frac{1}{0.72}\right)}$$

$$= \frac{-0.28125}{(Z - 0.72)(Z - 1.3889)}$$

Now the poles of $H(Z)H(Z^{-1})Z^{-1}$ are $p_1=0.72$, $p_2=1.3889$

Output noise power due to input quantization

$$\sigma_{eo}^2 = \sigma_e^2 \left[\frac{1}{2\pi j} \oint H(Z)H(Z^{-1})Z^{-1} dZ \right]$$

$$= \sigma_e^2 \sum_{i=1}^{N} \operatorname{Re} s\left[H(Z)H(Z^{-1})Z^{-1}\right]\Big|_{z=p_i}$$

$$= \sigma_e^2 \sum_{i=1}^{N} \operatorname{Re} s\left[H(Z)H(Z^{-1})Z^{-1}\right]\Big|_{z=p_i}$$

Where $p_1, p_2, \ldots p_n$ are the poles of $H(Z)H(Z^{-1}) Z^{-1}$ that lies inside the unit circle in z-plane.

$$\sigma_{eo}^2 = \sigma_e^2 \times (Z - 0.72) \times \frac{-0.28125}{(Z - 0.72)(Z - 1.3889)}\Bigg|_{Z=0.72}$$

$$= \sigma_e^2 \times \frac{-0.28125}{0.72 - 1.3889}$$

$$= 0.4205\sigma_e^2$$

****

Consider the transfer function $H(z) = H_1(z)H_2(z)$ **where** $H_1(z) = \frac{1}{1 - a_1 z^{-1}}$ *and* $H_2(z) = \frac{1}{1 - a_2 z^{-1}}$

Find the output round off noise power. Assume $\alpha_1 = 0.5 \, and \, \alpha_2 = 0.6$ and find output round off noise power.

Solution:

The round off noise model for $H(z) = H_1(z)H_2(z)$ is given by,

From the realization we can find that the noise transfer function seen by noise source $e_1(n)$ is H(z), where,

$$H(z) = \frac{1}{(1-a_1z^{-1})(1-a_2z^{-1})} \text{------------------------}(1)$$

Whereas, the noise transfer function seen by $e_2(n)$ *is,*

$$H_2(z) = \frac{1}{(1-a_2z^{-1})} \text{------------------------------}(2)$$

The total steady state noise variance can be obtained, we have

$$\sigma_0^{\ 2} = \sigma_{01}^{\ 2} + \sigma_{02}^{\ 2} \text{------------------------------}(3)$$

$$\sigma_{01}^{\ 2} = \frac{1}{2\pi j}\oint_c H(z)H(z^{-1})z^{-1}dz$$

$$= \sigma_e^{\ 2}\frac{1}{2\pi j}\oint_c \frac{1}{1-a_1z^{-1}}\frac{1}{1-a_2z^{-1}}\frac{1}{1-a_1z}\frac{1}{1-a_2z}z^{-1}dz$$

$$= \sigma_e^{\ 2}\left[\sum of \ residue \ of \ H(z)H(z^{-1})z^{-1} \ at \ po1es \ z=a_1, z=a_2, z=\frac{1}{a_1} \ and \ z=\frac{1}{a_2}\right]$$

If a1 and a2 are less than the poles $z=1/a_1$ and $z=1/a_2$ lies outside of the circle $|z|=1$. So, the residue of H(z) $H(z^{-1})$ z^{-1}at $z=1/a_1$ and $z=1/a_2$ are zero. Consequently we have,

$$\sigma_{01}^{\ 2} = \left[\sum of \ residue \ of \ H(z)H(z^{-1})z^{-1} \ at \ po1es \ z=a_1, z=a_2\right]$$

$$= \left[(z-a_1)\frac{z^{-1}}{(1-a_1z^{-1})(1-a_2z^{-1})(1-a_1z)(1-a_2z)}\bigg|_{z=a_1} + (z-a_2)\frac{z^{-1}}{(1-a_1z^{-1})(1-a_2z^{-1})(1-a_1z)(1-a_2z)}\bigg|_{z=a_2}\right]$$

$$= \sigma_e^{\ 2}\left[\frac{1}{\left(1-\frac{a_2}{a_1}\right)(1-a_2^{\ 2})(1-a_1a_2)} + \frac{1}{\left(1-\frac{a_2}{a_1}\right)(1-a_1a_2)(1-a_2^{\ 2})}\right]$$

$$\sigma_{01}^{\ 2} = \sigma_e^{\ 2}\left[\frac{a_1}{a_1-a_2}\cdot\frac{1}{1-a_1^{\ 2}}\cdot\frac{1}{1-a_1a_2} + \frac{a_2}{a_2-a_1}\cdot\frac{1}{1-a_2^{\ 2}}\cdot\frac{1}{1-a_1a_2}\right] \text{--------------}(4)$$

In the same way,

$$\sigma_{02}^{\ 2} = \frac{\sigma_e^{\ 2}}{2\pi j}\oint_c H_2(z)H_2(z^{-1})z^{-1}dz$$

$$= \frac{\sigma_e^{\ 2}}{2\pi j}\oint_c \frac{1}{1-a_2z^{-1}}\frac{1}{1-a_2z}z^{-1}dz$$

$$= \sigma_e^{\ 2}\left[(z-a_2)\frac{z^{-1}}{(1-a_2z^{-1})(1-a_2z)}\bigg|_{z=a_2}\right]$$

$$= \sigma_e^{\ 2}\left[(z-a_2z^{-1})\frac{z^{-1}}{(1-a_2z^{-1})(1-a_2z)}\bigg|_{z=a_2}\right]$$

$$= \sigma_e^{\ 2}\left[\frac{1}{1-a_2^{\ 2}}\right] \text{------------------------------------}(5)$$

$$\sigma_0^{\ 2} = \sigma_e^{\ 2}\left[\frac{1}{1-a_2^{\ 2}} + \frac{a_1}{a_1-a_2}\cdot\frac{1}{1-a_1^{\ 2}}\cdot\frac{1}{1-a_1a_2} + \frac{a_2}{a_2-a_1}\cdot\frac{1}{1-a_2^{\ 2}}\cdot\frac{1}{1-a_1a_2}\right]$$

$$= \sigma_e^{\ 2}\left[\frac{1}{1-a_2^{\ 2}} + \frac{a_1\left(1-a_2^{\ 2}\right) - a_2^{\ 2}\left(1-a_1^{\ 2}\right)}{\left(1-a_1^{\ 2}\right)\left(1-a_2^{\ 2}\right)\left(1-a_1a_2\right)\left(a_1-a_2\right)}\right]$$

$$= \sigma_e^{\ 2}\left[\frac{1}{1-a_2^{\ 2}} + \frac{\left(a_1-a_2\right)\left(1+a_1a_2\right)}{\left(1-a_1^{\ 2}\right)\left(1-a_2^{\ 2}\right)\left(1-a_1a_2\right)\left(a_1-a_2\right)}\right]$$

$$= \frac{2^{-2b}}{12}\left[\frac{1}{1-a_2^{\ 2}} + \frac{\left(1+a_1a_2\right)}{\left(1-a_1^{\ 2}\right)\left(1-a_2^{\ 2}\right)\left(1-a_1a_2\right)}\right]$$

The steady state noise power for $a_1 = 0.5, a_2 = 0.6$ is given by

$$= \frac{2^{-2b}}{12}\left[\frac{1}{1-(0.6)^2} + \frac{1+(0.5)(0.6)}{\left(1-(0.5)^2\right)\left(1-(0.6)^2\right)\left(1-0.6(0.5)\right)}\right]$$

$$= \frac{2^{-2b}}{12}(5.4315)$$

****

Draw the quantization noise model for a second order system $H(z) = \dfrac{1}{1-2r\cos\theta z^{-1} + r^2z^{-2}}$ **and find the steady state output noise variance.**

Solution:

Given:

$$H(z) = \frac{1}{1-2r\cos\theta z^{-1} + r^2z^{-2}}$$

The quantization noise model is,

we know, $\sigma_0^{\ 2} = \sigma_{01}^{\ 2} + \sigma_{02}^{\ 2}$

Both noise sources see the same transfer function

$$H(z) = \frac{1}{1-2r\cos\theta z^{-1} + r^2z^{-2}}$$

The impulse response of the transfer function is given by

$$h(n) = r^n\,\frac{\sin(n+1)\theta}{\sin\theta}\,u(n)$$

Now the steady state output noise variance is,

$$\sigma_0^{\ 2} = \sigma_{01}^{\ 2} + \sigma_{02}^{\ 2}$$

But $\sigma_{01}^{\ 2} = \sigma_{02}^{\ 2} = \sigma_e^{\ 2}\sum_{n=-\infty}^{\infty} h^2(n),$ *which gives us*

$$\sigma_0^{\ 2} = 2.\frac{2^{-2b}}{12}\sum_{n=0}^{\infty} r^{2n}\,\frac{\sin^2(n+1)\theta}{\sin^2\theta}$$

$$= 2.\frac{2^{-2b}}{12}\frac{1}{2\sin^2\theta}\sum_{n=0}^{\infty} r^{2n}\left[1-\cos 2(n+1)\theta\right] \qquad \therefore \cos 2\theta = 1-2\sin^2\theta$$

$$= \frac{2^{-2b}}{6}\frac{1}{2\sin^2\theta}\left[\sum_{n=0}^{\infty} r^{2n} - \sum_{n=0}^{\infty} r^{2n}\cos 2(n+1)\theta\right]$$

$$= \frac{2^{-2b}}{6}\frac{1}{2\sin^2\theta}\left[\frac{1}{1-r^2} - \frac{1}{2}\left(\sum_{n=0}^{\infty} r^{2n}e^{j2(n+1)\theta} + \sum_{n=0}^{\infty} r^{2n}e^{-j2(n+1)\theta}\right)\right]$$

$$= \frac{2^{-2b}}{6}\frac{1}{2\sin^2\theta}\left[\frac{1}{1-r^2} - \frac{1}{2}\left(\frac{e^{j2\theta}}{1-r^2e^{2j\theta}} + \frac{e^{-j2\theta}}{1-r^2e^{-2j\theta}}\right)\right]$$

$$= \frac{2^{-2b}}{6}\frac{1}{2\sin^2\theta}\left[\frac{1}{1-r^2} - \frac{\cos 2\theta - r^2}{1-2r^2\cos 2\theta + r^4}\right]$$

$$= \frac{2^{-2b}}{6}\frac{1}{2\sin^2\theta}\left[\frac{(1+r)^2(1-\cos 2\theta)}{(1-r^2)(1-2r^2\cos 2\theta + r^4)}\right]$$

$$= \frac{2^{-2b}}{6}\frac{(1+r)^2}{(1-r^2)(1-2r^2\cos 2\theta + r^4)}$$

****

Co-efficient quantization error

- We know that the IIR Filter is characterized by the system function

$$H(Z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1+\sum_{k=1}^{N} a_k z^{-k}}$$

- After quantizing ,

$$[H(Z)]_q = \frac{\sum_{k=0}^{M} [b_k]_q z^{-k}}{1+\sum_{k=1}^{N} [a_k]_q z^{-k}}$$

Where $[a_k]_q = a_k + \Delta a_k$

$[b_k]_q = b_k + \Delta b_k$

- The quantization of filter coefficients alters the positions of the poles and zeros in z-plane.
  1. If the poles of desired filter lie close to the unit circle, then the quantized filter poles may lie outside the unit circle leading into instability of filter.
  2. Deviation in poles and zeros also lead to deviation in frequency response.

****

Consider a second order IIR filter with $H(z) = \frac{1.0}{(1-0.5z^{-1})(1-0.45z^{-1})}$ find the effect on quantization on pole locations of the given system function in direct form and in cascade form. Take b=3bits. [Apr/May-10] [Nov/Dec-11]

Solution:

Given that,

$$H(z) = \frac{1.0}{(1-0.5z^{-1})(1-0.45z^{-1})}$$

$$H(z) = \frac{1}{z^{-1}(z-0.5z^{-1})z^{-1}(z-0.5)}$$

$$= \frac{z^2}{(z-0.5)(z-0.45)}$$

The roots of the denominator of H(z) are the original poles of H(z). let the original poles of H(z) be p_1 and p_2.

Here $p_1=0.5$ and $p_2=0.45$

Direct form I:

$$H(z) = \frac{1.0}{(1-0.5z^{-1})(1-0.45z^{-1})}$$

$$H(z) = \frac{1}{1-0.5z^{-1}-0.45z^{-1}+0.225z-2}$$

$$= \frac{1}{1-0.95z^{-1}+0.225z^{-2}}$$

Let us quantize the coefficients by truction.

$.95_{10} \xrightarrow[\text{Binary}]{\text{Convert to}} .1111_2 \xrightarrow[\text{3-bits}]{\text{Truncate to}} .111_2 \xrightarrow[\text{decimal}]{\text{Convert to}} .875_{10}$

$.225_{10} \xrightarrow[\text{Binary}]{\text{Convert to}} .0011_2 \xrightarrow[\text{3-bits}]{\text{Truncate to}} .001_2 \xrightarrow[\text{decimal}]{\text{Convert to}} .125_{10}$

Let $\overline{H}(z)$ be the transfer function of the IIR system after quantizing the coefficients.

$$\overline{H}(z) = \frac{1}{1-0.875z^{-1}+0.125z^{-2}}$$

$$let\ \overline{H}(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-0.875z^{-1}+0.125z^{-2}}$$

On cross multiplying the above equation we get,

$$Y(z) - 0.875z^{-1}Y(z) + 0.125z^{-2}Y(z) = X(z)$$

$$Y(z) = X(z) + 0.875z^{-1}Y(z) - 0.125z^{-2}Y(z)$$

<u>Cascade form</u>:

Given that

$$H(z) = \frac{1.0}{(1-0.5z^{-1})(1-0.45z^{-1})}$$

In cascade realization the system can be realized as cascade of first order sections.

$H(z)=H_1(z)+H_2(z)$

Where,

$$H_1(z) = \frac{1}{1-0.5z^{-1}}\ and\ \ H_2(z) = \frac{1}{1-0.45z^{-1}}$$

Let us quantize the coefficients of $H_1(z)$ and $H_2(z)$ by truncation.

$.5_{10} \xrightarrow[\text{Binary}]{\text{Convert to}} .1000_2 \xrightarrow[\text{3-bits}]{\text{Truncate to}} .100_2 \xrightarrow[\text{decimal}]{\text{Convert to}} .5_{10}$

$.45_{10} \xrightarrow[\text{Binary}]{\text{Convert to}} .0111_2 \xrightarrow[\text{3-bits}]{\text{Convert to}} .011_2 \xrightarrow[\text{decimal}]{\text{Convert to}} .375_{10}$

let , $\overline{H_1}(z)$ and $\overline{H_2}(z)$ be the transfer function of the first-order sections after quantizing the coefficients.

$$\overline{H_1}(z) = \frac{1}{1-0.5z^{-1}}$$

$$\overline{H_2}(z) = \frac{1}{1-0.375z^{-1}}$$

$$let, \overline{H_1}(z) = \frac{Y_1(z)}{X(z)} = \frac{1}{1-0.5z^{-1}}$$

$$Y_1(z) - 0.5z^{-1}Y_1(z) = X(z)$$

$$Y_1(z) = X(z) + 0.5z^{-1}Y_1(z)$$

$$\text{let}, \overline{H_2}(z) = \frac{Y(z)}{Y_1(z)} = \frac{1}{1-0.375z^{-1}}$$

on cross multiplying the above equation we get,

$$Y(z) - 0.375z^{-1}Y(z) = Y_1(z)$$

$$Y(z) = Y_1(z) + 0.375z^{-1}Y(z)$$

****

Round off effects and overflow in digital filter:

***Explain in detail about round off effects in digital filters.**

- The presence of one or more quantizer in the realization of a digital filter results in a non-linear device. i.e. recursive digital filter may exhibit undesirable oscillations in its output
- In the finite arithmetic operations, some registers may overflow if the input signal level becomes large.
- These overflow represents non-linear distortion leading to limit cycle oscillations
- There are two types of limit cycle oscillations which includes
  1. Zero input limit cycle oscillations (Low amplitude compared to overflow limit cycle oscillations)
  2. Over flow limit cycle oscillations.

Zero input limit cycle oscillations

- The arithmetic operations produces oscillations even when the input is zero or some non zero constant values. Such oscillations are called zero input limit cycle oscillations.

Overflow limit cycle oscillations

- The limit cycle occurs due to the overflow of adder is known as overflow limit cycle oscillations.

Dead Band:

The limit cycle occurs as a result of quantization effect in multiplication. The amplitude of the output during a limit cycle is confined to a range of values called the dead band of the filter.

$$|y(n-1)| \le \frac{\frac{2^{-b}}{2}}{(1-|a|)}$$

Consider a first order filter

$$y(n) = ay(n-1) + x(n); \qquad n > 0$$

After rounding the product

$$y_q(n) = Q[a * y(n-1)] + x(n);$$

The round off error

$$-\frac{2^{-b}}{2} \le e_r \le \frac{2^{-b}}{2}$$

where, e_r →difference between the quantized value and the actual value.

$$Q[ay(n-1) - ay(n-1)] \le \frac{2^{-b}}{2}$$

The dead band of the filter for the limit cycle oscillations are

$$Q[ay(n-1)] = \begin{cases} y(n-1) & a > 0 \\ -y(n-1) & a < 0 \end{cases}$$

$$|y(n-1)| - a|y(n-1)| \le \frac{2^{-b}}{2}$$

$$y(n-1)\,(1-|a|) \le \frac{2^{-b}}{2}$$

Dead band of the filter, $|y(n-1)| \le \dfrac{\frac{2^{-b}}{2}}{(1-|a|)}$

****

Problem: Consider a 1st order FIR system equation $y(n) = x(n) + ay(n-1)$ **with**

$$x(n) = \begin{cases} 0.875 & , \; n = 0 \\ 0 & , \; otherwise \end{cases}$$

Find the limit cycle effect and the dead band. Assume b=4 and a=0.95. (Nov/Dec-12)(Nov/Dec-15) [May/June-2016]

Solution:

Given:

$$x(n) = \begin{cases} 0.875 & , \; n = 0 \\ 0 & , \; otherwise \end{cases}$$

$$Dead\ band = \frac{2^{-b}}{2(1-|a|)} = \frac{2^{-4}}{2(1-|0.95|)} = 0.625$$

$$y(n) = x(n) + 0.95y(n-1)$$

n	***x(n)***	$y(n-1)$	$ay(n-1)$	$Q[ay(n-1)]$ *(round off to 4-bits)*	$y(n) = x(n) + Q[ay(n-1)]$
0	0.875	0	0	0.0000	y(0)=0.875
1	0	0.875	$0.875*0.95$ $=(0.83125)_{10}$ $=(0.11010)_2$	$=(0.1101)_2$ $=0.8125$	y(1)=0.8125
2	0	0.8125	$0.8125*0.95$ $=(0.77187)_{10}$ $=(0.110001)_2$	$=(0.1100)_2$ $=0.75$	$y(2)=0.75$
3	0	0.75	$0.75*0.95$ $=(0.7125)_{10}$ $=(0.1011011)_2$	$=(0.1011)_2$ $=0.6875$	$y(3)=0.6875$
4	0	0.6875	$0.6875*0.95$ $=(0.653125)_{10}$ $=(0.101001)_2$	$=(0.1010)_2$ $=0.625$	$y(4)=0.625$
5	0	0.625	$0.625*0.95$ $=(0.59375)_{10}$ $=(0.10011)_2$	$=(0.1010)_2$ $=0.625$	$y(5)=0.625$
6	0	0.625	$0.625*0.95$ $=(0.59375)_{10}$ $=(0.10011)_2$	$=(0.1010)_2$ $=0.625$	$y(6)=0.625$

Conclusion:

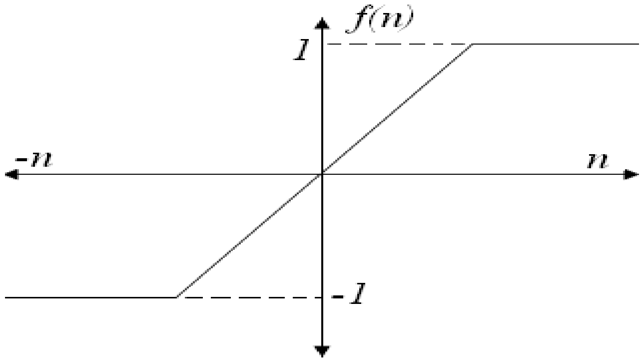
The dead band of the filter is 0.625. When $n \geq 5$ the output remains constant at 0.625 causing limit cycle oscillations.

****

Overflow Limit cycle oscillations:

***What are called overflow oscillations? How it can be prevented?**

- We know that the limit cycle oscillation is caused by rounding the result of multiplication.
- The limit cycle occurs due to the overflow of adder is known as overflow limit cycle oscillations.\
- Several types of limit cycle oscillations are caused by addition, which makes the filter output oscilate between maximum and minimum amplitudes.
- Let us consider 2 positive numbers n_1 & n_2
  n_1=0.111→7/8
  n_2=0.110→6/8
  n_1 + n2=1.101→-5/8 in sign magnitude form.
  The sum is wrongly interpreted as a negative number.
- The transfer characteristics of an saturation adder is shown in fig below
  where n → The input to the adder
  f(n) → The corresponding output



Saturation adder transfer characteristics

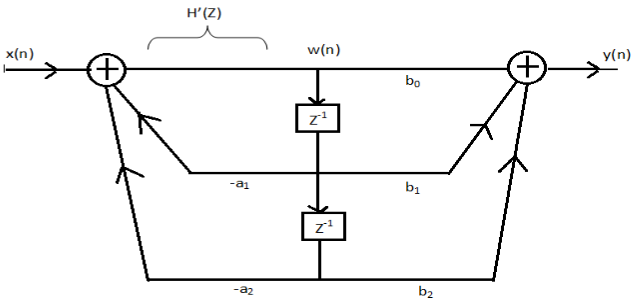
- From the transfer characteristics, we find that when overflow occurs, the sum of adder is set equal to the maximum value.

****

Signal Scaling:

***Explain how reduction of round-off errors is achieved in digital filters. [Nov/Dec-2016]**

- Saturation arithmetic eliminates limit cycles due to overflow, but it causes undeniable signal distortion due to the non linearity of the clipper.
- In order to limit the amount of non linear distortion, it is important to scale input signal and unit sample response between input and any internal summing node in the system to avoid overflow.



Realization of a second order IIR Filter

- Let us consider a second order IIR filter as shown in the above figure. Here a scale factor S_0 is introduced between the input x(n) and the adder 1 to prevent overflow at the output of adder 1.

- Now the overall input-output transfer function is

Now the transfer function

$$H(z) = S_0 \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

$$= S_0 \frac{N(z)}{D(z)}$$

From figure

$$H^{'}(z) = \frac{W(z)}{X(z)} = \frac{S_0}{1 + a_1 z^{-1} + a_2 z^{-2}} = \frac{S_0}{D(z)}$$

$$W(z) = \frac{S_0 X(z)}{D(z)} = S_0 \, \mathrm{S}(z)\, \mathrm{x}(\mathrm{Z})$$

Where $S(z) = \dfrac{1}{D(z)}$

we have

$$w(n) = \frac{S_0}{2\pi} \int S(e^{j\theta}) X(e^{j\theta})(e^{jn\theta})\, \mathrm{d}\theta$$

$$\mathrm{w(n)}\ ^2 = \frac{S_0^2}{2\pi^2} \left| \int S(e^{j\theta}) X(e^{j\theta})(e^{jn\theta})\, \mathrm{d}\theta \right|^2$$

Using Schwartz inequality

$$\mathrm{w(n)}\ ^2 \le S_0^2 \left[\int_{2\pi} \left| S(e^{j\theta}) \right|^2 \mathrm{d}\theta \right] \left[\int_{2\pi} \left| X(e^{j\theta}) \right|^2 \mathrm{d}\theta \right]$$

Applying parsevals theorem

$$\mathrm{w(n)}\ ^2 \le S_0^2 \sum_{n=0}^{\infty} x^2(n) \frac{1}{2\pi} \int_{2\pi} \left| S(e^{j\theta}) \right|^2 \mathrm{d}\theta$$

$$if \ \ z = e^{j\theta} \ \ then \ dz = je^{j\theta}\, d\theta$$

which gives

$$d\theta = \frac{dz}{jz}$$

By substituting all values

$$\mathrm{w(n)}\ ^2 \le S_0^2 \sum_{n=0}^{\infty} x^2(n) \frac{1}{2\pi j} \int_c \left| S(z) \right|^2 \mathrm{z}^{-1} \mathrm{d\,z}$$

$$\mathrm{w(n)}\ ^2 \le S_0^2 \sum_{n=0}^{\infty} x^2(n) \frac{1}{2\pi j} \int_c S(z) S(\mathrm{z}^{-1})\, \mathrm{z}^{-1} \mathrm{d\,z}$$

$$w^2(n) \le \sum_{n=0}^{\infty} x^2(n) \ \ when$$

$$S_0^2 \frac{1}{2\pi j} \int_c S(z) S(z^{-1})\, \mathrm{d\,z} = 1$$

Which gives us,

$$S_0^2 = \frac{1}{\frac{1}{2\pi j}\int_c S(z)S(z^{-1})\,z^{-1}dz}$$

$$= \frac{1}{\frac{1}{2\pi j}\int_c \frac{z^{-1}dz}{D(z)D(z^{-1})}}$$

$$S_0^2 = \frac{1}{I}$$

Where I=

$$\frac{1}{2\pi j}\int_c \frac{z^{-1}dz}{D(z)D(z^{-1})}$$

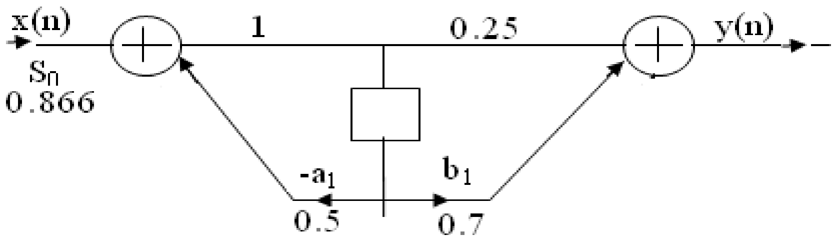
Note:

- Because of the process of scaling, the overflow is eliminated. Here so is the scaling factor for the first stage.
- Scaling factor for the second stage = S_{01} and it is given by $S_{01}^2 = \frac{1}{S_0^2 I_2}$

  Where $I_2 = \frac{1}{2\pi j}\oint_c \frac{H_1(Z)H_1(Z^{-1})Z^{-1}}{D_2(Z)D_2(Z^{-1})}dZ$

  ****

For the given transfer function, $H(Z) = \frac{0.25 + 0.7Z^{-1}}{1 - 0.5Z^{-1}}$**, find scaling factor so as to avoid overflow in the adder '1' of the filter.**



Given:

$D(Z) = 1-0.5Z^{-1}$

$D(Z^{-1}) = 1-0.5\,Z$

Solution:

$$I = \frac{1}{2\pi j}\oint_c \frac{1}{D(Z)D(Z^{-1})}\frac{dZ}{Z}$$

$$= \frac{1}{2\pi j}\oint_c \frac{1}{(1-0.5Z^{-1})(1-0.5Z)}\frac{dZ}{Z}$$

$$= \frac{1}{2\pi j}\oint_c \frac{Z}{(Z-0.5)}\frac{1}{(1-0.5Z)}\frac{dZ}{Z}$$

Residue of $\left.\frac{Z}{(Z-0.5)}\frac{1}{(1-0.5Z)}\right|_{Z=0.5} + 0$

I=1.3333

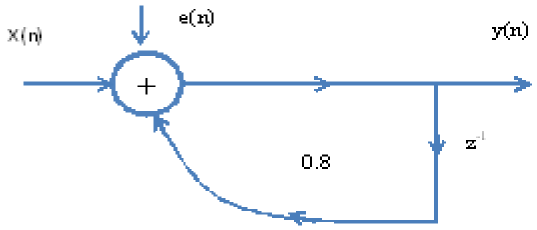
$$S_0 = \frac{1}{\sqrt{I}}$$

$$S_0 = \frac{1}{\sqrt{1.333}}$$

$$= 0.866$$

****

Consider the recursive filter shown in fig. The input x(n) has a range of values of ±100V, represented by 8 bits. Compute the variance of output due to A/D conversion process. (6)



Solution:

Given the range is ±100V

The difference equation of the system is given by $y(n) = 0.8y(n-1) + x(n)$, whose impulse response h(n) can be obtained as

$$h(n) = (0.8)^n u(n)$$

$$quantization\ step\ size = \frac{range\ of\ the\ signal}{No.of\ quantization\ levels}$$

$$= \frac{200}{2^8}$$

$$= 0.78125$$

$Variance\ of\ the\ error\ signal$

$$\sigma_e^2 = \frac{q^2}{12}$$

$$= \frac{(0.78125)^2}{12}$$

$$\sigma_e^2 = 0.05086$$

$Variance\ of\ output$

$$\sigma_{\not\in}^2 = \sigma_e^2 \sum_{n=0}^{\alpha} h^2(n)$$

$$= (0.05086)\sum_{n=0}^{\alpha}(0.8)^{2n}$$

$$= \frac{0.05086}{1-(0.8)^2} = 0.14128$$

****

The input to the system y(n)=0.999y(n-1)+x(n) is applied to an ADC. What is the power produced by the quantization noise at the output of the filter if the input is quantized to a) 8 bits b)16 bits. May-07

Solution:

y(n)=0.999y(n-1)+x(n)

Taking z-transform on both sides

$Y(z)=0.999z^{-1}Y(z)+X(z)$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-0.999z^{-1}}$$

$$H(z)H(z^{-1})z^{-1} = (\frac{z}{z-0.999})(\frac{z^{-1}}{z^{-1}-0.999})z^{-1}$$

$$= \frac{z^{-1}}{(z-0.999)(-0.999)(z-\frac{1}{0.999})}$$

$$= \frac{-0.001}{(z-0.999)(z-0.001)}$$

$$\left.\begin{matrix}\textit{output noise power due}\\ \textit{to input quantization}\end{matrix}\right\}\sigma_{eoi}^2 = \sigma_e^2 \frac{1}{2\pi j}\int_c H(z)H(z^{-1})z^{-1}dz$$

$$= \sigma_e^2 \sum_{i=1}^{N} \mathrm{Re}\, s\left[H(z)H(z^{-1})z^{-1}\right]\Big|_{z=p_i}$$

$$= \sigma_e^2 \sum_{i=1}^{N} \left[(z = p_i)H(z)H(z^{-1})z^{-1}\right]\Big|_{z=p_i}$$

Where $p_1, p_2, \ldots\ldots p_N$ are poles of $H(z)H(z^{-1})z^{-1}$, that lies inside the unit circle in z-plane.

$$\sigma_{eoi^2} = \sigma_{e^2}(z-0.999)(\frac{0.001}{(z-0.999)(z-0.001)})\Big|_{z=0.999}$$

$$= \sigma_{e^2}500.25$$

a) b+1=8 bits(Assuming including sign bit)

$$\sigma_\varepsilon^2 = \frac{2^{2(7)}}{12}(500.25) = 2.544\times10^{-3}$$

b) b+1=16 bits

$$\sigma_\varepsilon^2 = \frac{2^{2(15)}}{12}(500.25) = 3.882\times10^{-8}$$

****

Find the effect of coefficient quantization on pole locations of the given second order IIR system, when it is realized in direct form I and in cascade form. Assume a word length of 4 bits through truncation.

$$H(z) = \frac{1}{1-0.9z^{-1}+0.2z^{-2}}$$

Solution:

<u>Direct form I</u>

Let b=4 bits including a sign bit

$(0.9)_{10} = (0.111001...)_2$

Integer part

$\frac{0.9\times 2}{1.8}$

$\mapsto$ 1 $\downarrow$

$\frac{0.8\times 2}{1.6}$

$\mapsto$ 1

$\frac{0.6\times 2}{1.2}$ 1

$\mapsto$

$\frac{0.2\times 2}{0.4}$

$\mapsto$ 0

$\frac{0.4\times 2}{0.8}$

$\mapsto$ 0

$\frac{0.8\times 2}{1.6}$

$\mapsto$ 1

After truncation we get

$(0.111)_2=(0.875)_{10}$

$(0.2)_{10}=(0.00110\ldots)_2$

$(0.2)_{10} = \frac{0.2\times 2}{0.4}$

$\mapsto$ 0 $\downarrow$

$\frac{0.4\times 2}{0.8}$

$\mapsto$ 0

$\frac{0.8\times 2}{1.6}$

$\mapsto$ 1

$\frac{0.6\times 2}{1.2}$

$\mapsto$ 1

$\frac{0.2\times 2}{0.4}$

$\mapsto$ 0

After truncation we get

$(0.001)_2=(0.125)_{10}$

The system function after coefficient quantization is

$$H(z) = \frac{1}{1-0.875z^{-1}+0.125z^{-2}}$$

Now the pole locations are given by

$z_1=0.695$

$z_2=0.178$

If we compare the Poles of H(z) and $\overline{H}(z)$ we can observe that the poles of $\overline{H}(z)$ deviate very much from the original poles.

Cascade form

$$H(z) = \frac{1}{1-0.5z^{-1}(1-0.4z^{-1})}$$

$(0.5)_{10} = (0.1000)_2$

After truncation we get

$(0.100)_2=(0.5)_{10}$
After truncation we get
$(0.011)_2=(0.375)_{10}$

$$(0.4)_{10} = \frac{0.4\times 2}{0.8} \mapsto 0 \downarrow$$
$$\frac{0.8\times 2}{1.6} \mapsto 0$$
$$\frac{0.6\times 2}{1.2} \mapsto 1$$
$$\frac{0.2\times 2}{0.4} \mapsto 1$$
$$\frac{0.4\times 2}{0.8} \mapsto 0$$

$(0.4)_{10}=(0.01100\ldots)_2$
The system function after coefficient quantization is

$$H(z) = \frac{1}{(1-0.5z^{-1})(1-0.375z^{-1})}$$

The pole locations are given by
$z_1=0.5$
$z_2=0.375$
on comparing the poles of the cascade system with original poles we can say that one of the poles is same and other pole is very close to original pole.
****

A LTI system is characterized by the difference equation y(n)=0.68y(n-1)+0.5x(n). The input signal x(n) has a range of -5V to +5V, represented by 8-bits. Find the quantization step size, variance of the error signal and variance of the quantization noise at the output.
Solution:
Given
Range R=-5V to +5V = 5-(-5) =10
Size of binary, B= 8 bits (including sign bit)
Quantization step size,

$$q = \frac{R}{2^8} = \frac{10}{2^8} = 0.0390625$$

$$\text{var}iance\,of\ error\ signal, \sigma_e^2 = \frac{q^2}{12} = \frac{0.0390625^2}{12} = 1.27116*10^{-4}$$

The difference equation governing the LTI system is
Y (n) =0.68y (n-1) +0.15x (n)
On taking z transform of above equation we get

$$Y(z) = 0.68z^{-1}Y(z) + 0.15X(z)$$
$$Y(z) - 0.68z^{-1}Y(z) = 0.15X(z)$$
$$Y(z)[1-0.68z^{-1}] = 0.15X(z)$$
$$\frac{Y(z)}{X(z)} = \frac{0.15}{1-0.68z^{-1}}$$
$$H(z) = \frac{Y(z)}{X(z)} = \frac{0.15}{1-0.68z^{-1}}$$
$$H(z)H(z^{-1})z^{-1} = \frac{0.15}{1-0.68z^{-1}} * \frac{0.15}{1-0.68z} * z^{-1}$$
$$H(z)H(z^{-1})z^{-1} = \frac{0.225z^{-1}}{\left(1-\frac{0.68}{z}\right)(-0.68)\left(z-\frac{1}{0.68}\right)}$$
$$H(z)H(z^{-1})z^{-1} = \frac{-0.0331z^{-1}}{\left(\frac{z-0.68}{z}\right)(z-1.4706)} = \frac{-0.0331z^{-1}}{(z-0.68)(z-1.4706)}$$

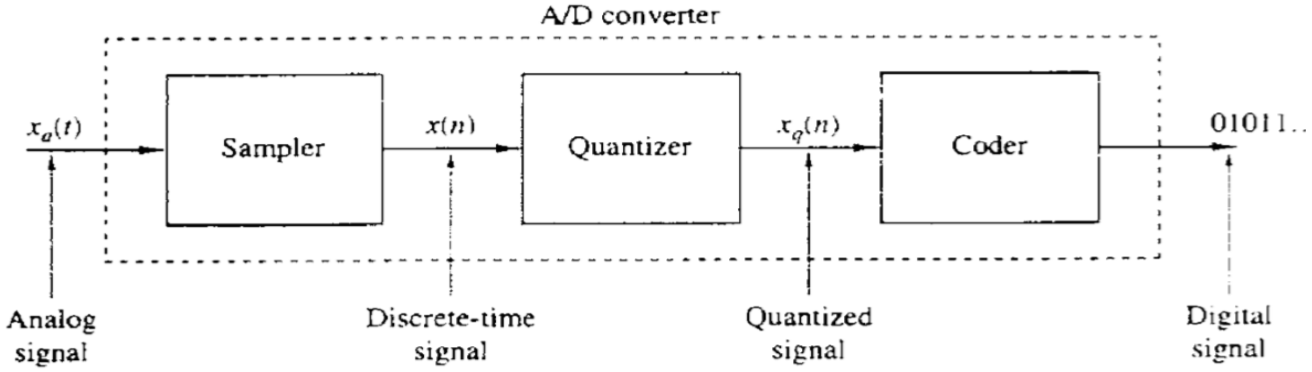
Now, poles of H (z) H (z^{-1}) z^{-1} are p_1=0.68, p_2=1.4706
Here, p_1=0.68 is the only pole that lies inside the unit circle in z-plane
Variance of the input quantization noise at the output.

$$\sigma_{eoi}^2 = \sigma_e^2 \frac{1}{2\pi j} \oint_c H(z)H(z^{-1})z^{-1}dz$$
$$\sigma_{eoi}^2 = \sigma_e^2 \sum_{i=1}^{N} \left[\text{Res}\, H(z)H(z^{-1})z^{-1} \right]\Big|_{z=pi}$$
$$\sigma_{eoi}^2 = \sigma_e^2 \sum_{i=1}^{N} \left[(z-p_i)H(z)H(z^{-1})z^{-1} \right]\Big|_{z=pi}$$
$$\sigma_{eoi}^2 = \sigma_e^2 (z-0.68) * \frac{-0.0331}{(z-0.68)(z-1.4706)}\Bigg|_{z=0.68}$$
$$\sigma_{eoi}^2 = \sigma_e^2 * \frac{-0.0331}{(0.68-1.4706)}\Bigg| = 0.0419\sigma_e^2$$
$$\sigma_{eoi}^2 = 0.0419 * 1.2716 * 10^{-4}$$
$$\sigma_{eoi}^2 = 5.328 * 10^{-6}$$

Analog to digital conversion:

10. Explain the ADC and DAC in detail.

A/D conversion has three process.



Basic parts of an analog-to digital (A/D) converter

1. Sampling

- Sampling is the conversion of a continuous- tome signal into a discrete-time signal obtained by taking the samples of continuous-time signal at discrete instants.
- Thus if $x_a(t)$ is the input to the sampler, the output is $x_a(nT) \equiv x(n)$, where T is called the sampling interval.

2. Quantisation

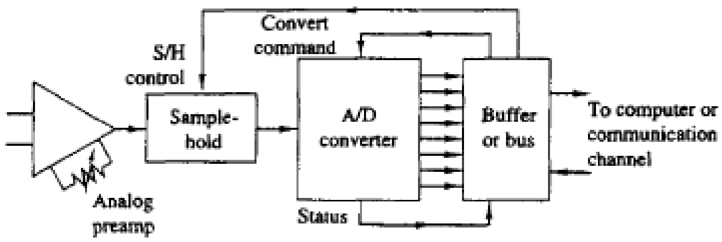
- The process of converting a discrete-time continuous amplitude signal into digital signal is called quantization.
- The value of each signal sample is represented by a value selected from a finite set of possible values.
- The difference between the unquantised sample x(n) and the quantized output $x_q(n)$ is called the quantization error or quantization noise.

$$e_q(n) = x_q(n) - x(n)$$

- To eliminate the excess bits either discard them by the process of truncation or discard them by rounding the resulting number by the process of rounding.
- The values allowed in the digital signals are called the quantization levels
- The distance Δ between two successive quantization levels is called the quantization step size or resolution.
- The quality of the output of the A/D converter is measured by the signal-to-quantization noise ratio.

3. Coding

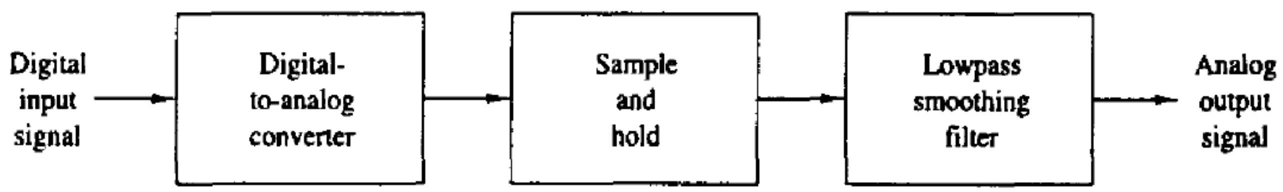
- In the coding process, each discrete value $x_q(n)$ is represented by a b-bit binary sequence.



Block diagram of basic elements of an A/D Converter

Digital to analog conversion:

- To convert a digital signal into an analog signal, digital to analog converters are used.



Basic operations in converting a digital signal into an analog signal

- The D/A converter accepts, at its input, electrical signals that corresponds to a binary word, and produces an output voltage or current that is proportional to the value of the binary word.
- The task of D/A converter is to interpolate between samples.
- The sampling theorem specifies the optimum interpolation for a band limited signal.
- The simplest D/A converter is the zero order hold which holds constant value of sample until the next one is received.
- Additional improvement can be obtained by using linear interpolation to connect successive samples with straight line segment.
- Better interpolation can be achieved y using more sophisticated higher order interpolation techniques.
- Suboptimum interpolation techniques result in passing frequencies above the folding frequency. Such frequency components are undesirable and are removed by passing the output of the interpolator through a proper analog filter which is called as post filter or smoothing filter.
- Thus D/A conversion usually involve a suboptimum interpolator followed by a post filter.

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